Chapter 5: Quantum Electrodynamics

This chapter gives an introduction to **quantum electrodynamics**, the quantum theory of the electromagnetic field and its interactions with electrons and other charged particles. We begin by formulating a quantum Hamiltonian for an electron in a classical electromagnetic field. Then we study how to quantize Maxwell's equations, arriving at a quantum field theory in which the elementary excitations are photons—particles of light. The final step is to formulate a theory in which electrons and photons are treated on the same quantum mechanical footing, as excitations of underlying quantum fields. Along the way, we will see how relativity can be accomodated with quantum theory.

Quantum electrodynamics is an extremely rich and intricate theory, and we will leave out many important topics. Interested readers are referred to are Dyson's 1951 lecture notes on quantum electrodynamics [1], and Zee's textbook *Quantum Field Theory in a Nutshell* [2].

I. QUANTIZATION OF THE LORENTZ FORCE LAW

A. Non-relativistic electrons in an electromagnetic field

Consider a non-relativistic charged particle in an electromagnetic field. As we are mainly interested in the physics of electrons interacting with electromagnetic fields, we henceforth take the electric charge of the particle to be $-e$, where $e = 1.602 \times 10^{-19}\,\mathrm{C}$ is the elementary charge. To describe particles with an arbitrary electric charge q, simply perform the substitution $e \to -q$ in the formulas you will subsequently encounter.

We wish to formulate the Hamiltonian governing the quantum dynamics of such a particle, subject to two simplifying assumptions: (i) the particle has charge and mass but is otherwise "featureless" (i.e., we ignore the spin angular momentum and magnetic dipole moment that real electrons possess), and (ii) the electromagnetic field is treated as a classical field, meaning that the electric and magnetic fields are definite quantities rather than operators. (We will see how to go beyond these simplifications later.)

Classically, the electromagnetic field acts on the particle via the Lorentz force law,

$$\mathbf{F}(\mathbf{r},t) = -e\Big(\mathbf{E}(\mathbf{r},t) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r},t)\Big), \tag{5.1}$$

where $\mathbf{r}$ and $\dot{\mathbf{r}}$ denote the position and velocity of the particle, t is the time, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. If no other forces are present, Newton's second law yields the equation of motion

$$m\ddot{\mathbf{r}} = -e\Big(\mathbf{E}(\mathbf{r},t) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r},t)\Big), \tag{5.2}$$

where m is the particle's mass. To quantize this, we must first convert the equation of motion into the form of Hamilton's equations of motion.

Let us introduce the electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}}{\partial t}, \tag{5.3}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t). \tag{5.4}$$

We now postulate that the equation of motion (5.2) can be described by the Lagrangian

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\Big[\Phi(\mathbf{r}, t) - \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)\Big]. \tag{5.5}$$

This follows the usual prescription for the Lagrangian as kinetic energy minus potential energy, with $-e\Phi$ serving as the potential energy function, except for the $-e\dot{\mathbf{r}} \cdot \mathbf{A}$ term. To see if this Lagrangian works, plug it into the Euler-Lagrange equations

$$\frac{\partial L}{\partial r_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i}. \tag{5.6}$$

The partial derivatives of the Lagrangian are:

$$\begin{aligned} \frac{\partial L}{\partial r_i} &= e\Big[\partial_i \Phi - \dot{r}_j\, \partial_i A_j\Big] \\ \frac{\partial L}{\partial \dot{r}_i} &= m\dot{r}_i - eA_i. \end{aligned} \tag{5.7}$$

Now we want to take the *total* time derivative of $\partial L/\partial \dot{r}_i$. In doing so, note that the $\mathbf{A}$ field has its own t-dependence, as well as varying with the particle's t-dependent position. Thus,

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} &= m\ddot{r}_i - e\,\frac{d}{dt}A_i(\mathbf{r}(t), t) \\ &= m\ddot{r}_i - e\,\partial_t A_i - e\,\dot{r}_j \partial_j A_i. \end{aligned} \tag{5.8}$$

(In the above equations, $\partial_i \equiv \partial/\partial r_i$, where r_i is the i-th component of the position vector, while $\partial_t \equiv \partial/\partial t$.) Plugging these expressions into the Euler-Lagrange equations (5.6) gives

$$\begin{aligned} m\ddot{r}_i &= -e\Big[\Big(-\partial_i \Phi - \partial_t A_i\Big) + \dot{r}_j\Big(\partial_i A_j - \partial_j A_i\Big)\Big] \\ &= -e\Big[E_i(\mathbf{r}, t) + \big(\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t)\big)_i\Big]. \end{aligned} \tag{5.9}$$

(The last step can be derived by expressing the cross product using the Levi-Cevita symbol, and using the identity $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.) This exactly matches Eq. (5.2), as desired.

We now use the Lagrangian to derive the Hamiltonian. The canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i - eA_i. \tag{5.10}$$

The Hamiltonian is defined as $H(\mathbf{r}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{r}} - L$. Using Eq. (5.10), we express it in terms of $\mathbf{p}$ rather than $\dot{\mathbf{r}}$:

$$\begin{aligned} H &= \mathbf{p} \cdot \left(\frac{\mathbf{p} + e\mathbf{A}}{m}\right) - \left(\frac{|\mathbf{p} + e\mathbf{A}|^2}{2m} + e\Phi - \frac{e}{m}(\mathbf{p} + e\mathbf{A}) \cdot \mathbf{A}\right) \\ &= \frac{|\mathbf{p} + e\mathbf{A}|^2}{m} - \frac{e}{m}\mathbf{A} \cdot (\mathbf{p} + e\mathbf{A}) - \left(\frac{|\mathbf{p} + e\mathbf{A}|^2}{2m} + e\Phi - \frac{e}{m}(\mathbf{p} + e\mathbf{A}) \cdot \mathbf{A}\right). \end{aligned} \tag{5.11}$$

After cancelling various terms, we obtain

$$H = \frac{|\mathbf{p} + e\mathbf{A}(\mathbf{r}, t)|^2}{2m} - e\Phi(\mathbf{r}, t). \tag{5.12}$$

This looks a lot like the Hamiltonian for a non-relativistic particle in a scalar potential,

$$H = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{r}, t).$$

In Eq. (5.12), the $-e\Phi$ term acts like a potential energy, which is no surprise. More interestingly, the vector potential appears via the substitution

$$\mathbf{p} \to \mathbf{p} + e\mathbf{A}(\mathbf{r}, t). \tag{5.13}$$

What does this mean? Think about what "momentum" means for a charged particle in an electromagnetic field. Noether's theorem states that each symmetry of a system (whether classical or quantum) is associated with a conservation law. Momentum is the quantity conserved when the system is symmetric under spatial translations. One of Hamilton's equations states that

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial r_i},$$

which implies that if H is $\mathbf{r}$-independent, then $d\mathbf{p}/dt = 0$. But when the electromagnetic potentials are $\mathbf{r}$-independent, the quantity $m\dot{\mathbf{r}}$ (which we usually call momentum) is not necessarily conserved! Take the potentials

$$\Phi(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = Ct\hat{z}, \tag{5.14}$$

where C is some constant. These potentials are $\mathbf{r}$-independent, but the vector potential is time-dependent, so the $-\dot{\mathbf{A}}$ term in Eq. (5.85) gives a non-vanishing electric field:

$$\mathbf{E}(\mathbf{r}, t) = -C\hat{z}, \quad \mathbf{B}(\mathbf{r}, t) = 0. \tag{5.15}$$

The Lorentz force law then says that

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = eC\hat{z}, \tag{5.16}$$

and thus $m\dot{\mathbf{r}}$ is not conserved. On the other hand, the quantity $\mathbf{p} = m\dot{\mathbf{r}} - e\mathbf{A}$ *is* conserved:

$$\frac{d}{dt}(m\dot{\mathbf{r}} - e\mathbf{A}) = eC\hat{z} - eC\hat{z} = 0. \tag{5.17}$$

Hence, this is the appropriate canonical momentum for a particle in an electromagnetic field.

We are now ready to go from classical to quantum mechanics. Replace $\mathbf{r}$ with the position operator $\hat{\mathbf{r}}$, and $\mathbf{p}$ with the momentum operator $\hat{\mathbf{p}}$. The resulting quantum Hamiltonian is

$$\hat{H}(t) = \frac{|\hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}}, t)|^2}{2m} - e\Phi(\hat{\mathbf{r}}, t). \tag{5.18}$$

Note: the momentum operator is $\hat{\mathbf{p}} = -i\hbar\nabla$ in the wavefunction representation, as usual.

B. Gauge symmetry

The Hamiltonian (5.18) possesses a subtle property known as **gauge symmetry**. Suppose we modify the scalar and vector potentials via the substitutions

$$\Phi(\mathbf{r},t) \to \Phi(\mathbf{r},t) - \dot{\Lambda}(\mathbf{r},t) \tag{5.19}$$

$$\mathbf{A}(\mathbf{r},t) \to \mathbf{A}(\mathbf{r},t) + \nabla\Lambda(\mathbf{r},t), \tag{5.20}$$

where $\Lambda(\mathbf{r},t)$ is an arbitrary scalar field called a **gauge field**. This is the **gauge transformation** of classical electromagnetism, which as we know leaves the electric and magnetic fields unchanged. When applied to the Hamiltonian (5.18), it generates a new Hamiltonian

$$\hat{H}_\Lambda(t) = \frac{|\hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}},t) + e\nabla\Lambda(\hat{\mathbf{r}},t)|^2}{2m} - e\Phi(\hat{\mathbf{r}},t) + e\dot{\Lambda}(\hat{\mathbf{r}},t). \tag{5.21}$$

Now suppose $\psi(\mathbf{r},t)$ is a wavefunction obeying the Schrödinger equation for the original Hamiltonian $\hat{H}$:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}(t)\psi(\mathbf{r},t) = \left[\frac{|\hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}},t)|^2}{2m} - e\Phi(\hat{\mathbf{r}},t)\right]\psi(\mathbf{r},t). \tag{5.22}$$

Then it can be shown that the wavefunction $\psi\,\exp(-ie\Lambda/\hbar)$ automatically satisfies the Schrödinger equation for the transformed Hamiltonian $\hat{H}_\Lambda$:

$$i\hbar\frac{\partial}{\partial t}\left[\psi(\mathbf{r},t)\,\exp\left(-\frac{ie\Lambda(\mathbf{r},t)}{\hbar}\right)\right] = \hat{H}_\Lambda(t)\left[\psi(\mathbf{r},t)\,\exp\left(-\frac{ie\Lambda(\mathbf{r},t)}{\hbar}\right)\right]. \tag{5.23}$$

To prove this, observe how time and space derivatives act on the new wavefunction:

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\psi\,\exp\left(-\frac{ie\Lambda}{\hbar}\right)\right] &= \left[\frac{\partial\psi}{\partial t} - \frac{ie}{\hbar}\dot{\Lambda}\,\psi\right]\exp\left(\frac{ie\Lambda}{\hbar}\right) \\
\nabla\left[\psi\,\exp\left(-\frac{ie\Lambda}{\hbar}\right)\right] &= \left[\nabla\psi - \frac{ie}{\hbar}\nabla\Lambda\,\psi\right]\exp\left(\frac{ie\Lambda}{\hbar}\right).
\end{aligned} \tag{5.24}$$

When the extra terms generated by the $\exp(ie\Lambda/\hbar)$ factor are slotted into the Schrödinger equation, they cancel the gauge terms in the scalar and vector potentials. For example,

$$\left(-i\hbar\nabla + e\mathbf{A} + e\nabla\Lambda\right)\left[\psi\,\exp\left(-\frac{ie\Lambda}{\hbar}\right)\right] = \left[\left(-i\hbar\nabla + e\mathbf{A}\right)\psi\right]\,\exp\left(-\frac{ie\Lambda}{\hbar}\right) \tag{5.25}$$

If we apply the $(-i\hbar\nabla + e\mathbf{A} + e\nabla\Lambda)$ operator a second time, it has a similar effect but with the quantity in square brackets on the right-hand side of (5.25) taking the place of ψ:

$$\left|-i\hbar\nabla + e\mathbf{A} + e\nabla\Lambda\right|^2\left[\psi\,\exp\left(-\frac{ie\Lambda}{\hbar}\right)\right] = \left[\left|-i\hbar\nabla + e\mathbf{A}\right|^2\psi\right]\,\exp\left(-\frac{ie\Lambda}{\hbar}\right). \tag{5.26}$$

The remainder of the proof for Eq. (5.23) can be carried out straightforwardly.

The above result can be stated in a simpler form if the electromagnetic fields are static. In this case, the time-independent electromagnetic Hamiltonian is

$$\hat{H} = \frac{|\hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}})|^2}{2m} - e\Phi(\hat{\mathbf{r}}). \tag{5.27}$$

Suppose $\hat{H}$ has eigenenergies $\{E_m\}$ and energy eigenfunctions $\{\psi_m(\mathbf{r})\}$. Then the gauge-transformed Hamiltonian

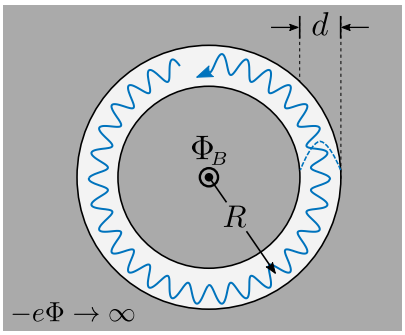
$$\hat{H}_\Lambda = \frac{|\hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}}) + e\nabla\Lambda(\mathbf{r})|^2}{2m} - e\Phi(\hat{\mathbf{r}}) \tag{5.28}$$

has the same energy spectrum $\{E_m\}$, with eigenfunctions $\{\,\psi_m(\mathbf{r})\exp[-ie\Lambda(\mathbf{r})/\hbar]\,\}$.

C. The Aharonov-Bohm effect

In quantum electrodynamics, it is the electromagnetic scalar and vector potentials that appear directly in the Hamiltonian, not the electric and magnetic fields. This has profound consequences. For example, even if a charged quantum particle resides in a region with zero magnetic field, it can feel the effect of nonzero *vector potentials* produced by magnetic fluxes elsewhere in space, a phenomenon called the **Aharonov-Bohm effect**.

A simple setting for observing the Aharonov-Bohm effect is shown in the figure below. A particle is trapped in a ring-shaped region (an "annulus"), of radius R and width $d \ll R$. Outside the annulus, we set $-e\Phi \to \infty$ so that the wavefunction vanishes; inside the annulus, we set $\Phi = 0$. We ignore the z-dependence of all fields and wavefunctions, so that the problem is two-dimensional. We define polar coordinates (r, ϕ) with the origin at the ring's center.



Now, suppose we thread magnetic flux (e.g., using a solenoid) through the origin, which lies in the region enclosed by the annulus. This flux can be described via the vector potential

$$\mathbf{A}(r, \phi) = \frac{\Phi_B}{2\pi r}\,\mathbf{e}_\phi, \tag{5.29}$$

where $\mathbf{e}_\phi$ is the unit vector pointing in the azimuthal direction. We can verify from Eq. (5.29) that the total magnetic flux through any loop of radius r enclosing the origin is $(\Phi_B/2\pi r)(2\pi r) = \Phi_B$. The fact that this is independent of r implies that the magnetic flux density is concentrated in an infintesimal area surrounding the origin, and zero everywhere else. However, the vector potential $\mathbf{A}$ is nonzero everywhere.

The time-independent Schrödinger equation is

$$\frac{1}{2m}\left|-i\hbar\nabla + \frac{e\Phi_B}{2\pi r}\,\mathbf{e}_\phi\right|^2 \psi(r, \phi) = E\psi(r, \phi), \tag{5.30}$$

with the boundary conditions $\psi(R \pm d/2, 0) = 0$. For sufficiently large R, we can guess that the eigenfunctions have the form

$$\psi(r,\phi) \approx \begin{cases} \psi_0 \cos\left(\frac{\pi}{d}(r-R)\right) e^{ikR\phi}, & r \in [R-d/2, R+d/2] \\ 0 & \text{otherwise.} \end{cases} \tag{5.31}$$

This describes a "waveguide mode" with a half-wavelength wave profile in the r direction (so as to vanish at $r = R \pm d/2$), traveling in the azimuthal direction with wavenumber k. The normalization constant ψ_0 is unimportant. We need the wavefunction to be single-valued under a 2π variation in the azimuthal coordinate, so

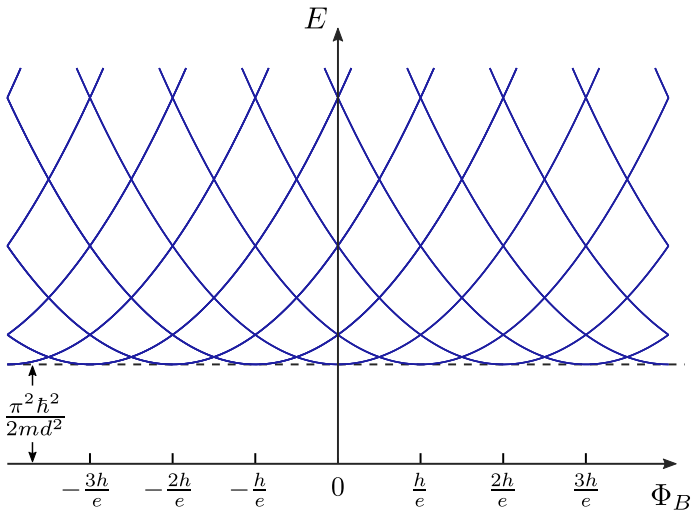
$$k \cdot 2\pi R = 2\pi n \quad \Rightarrow \quad k = \frac{n}{R}, \quad \text{where } n \in \mathbb{Z}. \tag{5.32}$$

Plugging this into Eq. (5.30) yields the energy levels

$$E_n = \frac{1}{2m}\left[\left(\frac{n\hbar}{R} + \frac{e\Phi_B}{2\pi R}\right)^2 + \left(\frac{\pi\hbar}{d}\right)^2\right] \tag{5.33}$$

$$= \frac{e^2}{8\pi^2 m R^2}\left(\Phi_B + \frac{nh}{e}\right)^2 + \frac{\pi^2\hbar^2}{2md^2}. \tag{5.34}$$

These energy levels are sketched versus the magnetic flux Φ_B in the figure below:



Each energy level has a quadratic dependence on Φ_B. Variations in Φ_B affect the energy levels despite the fact that $\mathbf{B} = 0$ in the annular region where the electron resides. This is a manifestation of the Aharonov-Bohm effect.

It is noteworthy that the curves of different n are centered at different values of Φ_B corresponding to multiples of $h/e = 4.13567 \times 10^{-5}\,\mathrm{T\,m^2}$, a fundamental unit of magnetic flux called the **magnetic flux quantum**. In other words, changing Φ_B by an exact multiple of h/e leaves the energy spectrum unchanged! This invariance property, which does not depend on the width of the annulus or any other geometrical parameters of the system, can be explained using gauge symmetry. When an extra flux of nh/e (where $n \in \mathbb{Z}$) is

threaded through the annulus, Eq. (5.29) tells us that the change in vector potential is $\Delta\mathbf{A} = (n\hbar/er)\mathbf{e}_\phi$. But we can undo the effects of this via the gauge field

$$\Lambda(r,\phi) = -\frac{n\hbar}{e}\,\phi \quad \Rightarrow \begin{cases} \nabla\Lambda & = -(n\hbar/er)\,\mathbf{e}_\phi \\ e^{-ie\Lambda/\hbar} & = e^{in\phi}. \end{cases} \tag{5.35}$$

Note that this Λ is not single-valued, but that's not a problem! Both $\nabla\Lambda$ and the phase factor $\exp(-ie\Lambda/\hbar)$ *are* single-valued, and those are the quantities that enter into the gauge symmetry relations (5.19)–(5.20).

II. DIRAC'S THEORY OF THE ELECTRON

A. The Dirac Hamiltonian

So far, we have been using $p^2/2m$-type Hamiltonians, which are limited to describing non-relativistic particles. In 1928, Paul Dirac formulated a Hamiltonian that can describe electrons moving close to the speed of light, thus successfully combining quantum theory with special relativity. Another triumph of Dirac's theory is that it accurately predicts the magnetic moment of the electron.

Dirac's theory begins from the time-dependent Schrödinger wave equation,

$$i\hbar\,\partial_t\,\psi(\mathbf{r},t) = \hat{H}\psi(\mathbf{r},t). \tag{5.36}$$

Note that the left side has a first-order time derivative. On the right, the Hamiltonian $\hat{H}$ contains spatial derivatives in the form of momentum operators. We know that time and space derivatives of wavefunctions are related to energy and momentum by

$$i\hbar\,\partial_t \;\leftrightarrow\; E, \qquad -i\hbar\,\partial_j \;\leftrightarrow\; p_j. \tag{5.37}$$

We also know that the energy and momentum of a relativistic particle are related by

$$E^2 = m^2c^4 + \sum_{j=1}^{3} p_j^2 c^2, \tag{5.38}$$

where m is the rest mass and c is the speed of light. Note that E and p appear to the same order in this equation. (Following the usual practice in relativity theory, we use Roman indices $j \in \{1,2,3\}$ for the spatial coordinates $\{x,y,z\}$.)

Since the left side of the Schrödinger equation (5.36) has a first-order time derivative, a relativistic Hamiltonian should involve first-order spatial derivatives. So we make the guess

$$\hat{H} = \alpha_0 mc^2 + \sum_{j=1}^{3} \alpha_j \hat{p}_j c, \tag{5.39}$$

where $\hat{p}_j \equiv -i\hbar\partial/\partial x_j$. The mc^2 and c factors are placed for later convenience. We now need to determine the dimensionless "coefficients" α_0, α_1, α_2, and α_3.

For a wavefunction with definite momentum $\mathbf{p}$ and energy E,

$$\hat{H}\psi = E\psi \quad \Rightarrow \quad \left(\alpha_0 mc^2 + \sum_{j=1}^{3} \alpha_j p_j c\right)\psi = E\,\psi. \tag{5.40}$$

This is obtained by replacing the $\hat{p}_j$ operators with definite numbers. If ψ is a scalar, this would imply that $\alpha_0 mc^2 + \sum_j \alpha_j p_j c = E$ for certain scalar coefficients $\{\alpha_0, \ldots, \alpha_3\}$, which does not match the relativistic energy-mass-momentum relation (5.38).

But we can get things to work if $\psi(\mathbf{r}, t)$ is a multi-component wavefunction, rather than a scalar wavefunction, and the α's are matrices acting on those components via the matrix-vector product operation. In that case,

$$\hat{H} = \hat{\alpha}_0 mc^2 + \sum_{j=1}^{3} \hat{\alpha}_j \hat{p}_j c, \quad \text{where} \quad \hat{p}_j \equiv -i\hbar\,\partial_j, \tag{5.41}$$

where the hats on $\{\hat{\alpha}_0, \ldots, \hat{\alpha}_3\}$ indicate that they are matrix-valued. Applying the Hamiltonian twice gives

$$\left(\hat{\alpha}_0 mc^2 + \sum_{j=1}^{3} \hat{\alpha}_j p_j c\right)^2 \psi = E^2\,\psi. \tag{5.42}$$

This can be satisfied if

$$\left(\hat{\alpha}_0 mc^2 + \sum_{j=1}^{3} \hat{\alpha}_j p_j c\right)^2 = E^2\,\hat{I}, \tag{5.43}$$

where $\hat{I}$ is the identity matrix. Expanding the square (and taking care of the fact that the $\hat{\alpha}_\mu$ matrices need not commute) yields

$$\hat{\alpha}_0^2 m^2 c^4 + \sum_j \left(\hat{\alpha}_0\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_0\right) mc^3 p_j + \sum_{jj'} \hat{\alpha}_j\hat{\alpha}_{j'}\, p_j p_{j'}\, c^2 = E^2\hat{I}. \tag{5.44}$$

This reduces to Eq. (5.38) if the $\hat{\alpha}_\mu$ matrices satisfy

$$\begin{aligned} \hat{\alpha}_\mu^2 &= \hat{I} \quad \text{for} \quad \mu = 0, 1, 2, 3, \quad \text{and} \\ \hat{\alpha}_\mu\hat{\alpha}_\nu + \hat{\alpha}_\nu\hat{\alpha}_\mu &= 0 \quad \text{for} \quad \mu \neq \nu. \end{aligned} \tag{5.45}$$

(We use Greek symbols for indices ranging over the four spacetime coordinates $\{0, 1, 2, 3\}$.) The above can be written more concisely using the anticommutator:

$$\{\hat{\alpha}_\mu, \hat{\alpha}_\nu\} = 2\delta_{\mu\nu}, \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3. \tag{5.46}$$

Also, we need the $\hat{\alpha}_\mu$ matrices to be Hermitian, so that $\hat{H}$ is Hermitian.

It turns out that the smallest possible Hermitian matrices that can satisfy Eq. (5.46) are 4×4 matrices. The choice of matrices (or "representation") is not uniquely determined.

One particularly useful choice is called the **Dirac representation**:

$$\hat{\alpha}_0 = \begin{bmatrix} \hat{I} & \hat{0} \\ \hat{0} & -\hat{I} \end{bmatrix}, \quad \hat{\alpha}_1 = \begin{bmatrix} \hat{0} & \hat{\sigma}_1 \\ \hat{\sigma}_1 & \hat{0} \end{bmatrix}$$
$$\hat{\alpha}_2 = \begin{bmatrix} \hat{0} & \hat{\sigma}_2 \\ \hat{\sigma}_2 & \hat{0} \end{bmatrix}, \quad \hat{\alpha}_3 = \begin{bmatrix} \hat{0} & \hat{\sigma}_3 \\ \hat{\sigma}_3 & \hat{0} \end{bmatrix}, \tag{5.47}$$

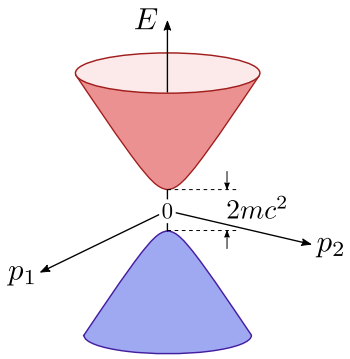
where $\{\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3\}$ denote the usual Pauli matrices. Since the $\hat{\alpha}_\mu$'s are 4×4 matrices, it follows that $\psi(\mathbf{r})$ is a four-component field.

B. Eigenstates of the Dirac Hamiltonian

According to Eq. (5.38), the energy eigenvalues of the Dirac Hamiltonian are

$$E = \pm\sqrt{m^2c^4 + \sum_j p_j^2 c^2}. \tag{5.48}$$

This is plotted below:



The energy spectrum forms two hyperbolic bands. For each $\mathbf{p}$, there are two degenerate positive energy eigenvalues, and two degenerate negative energy eigenvalues, for a total of four eigenvalues (matching the number of wavefunction components). The upper band matches the dispersion relation for a massive relativistic particle, as desired. But what about the negative-energy band? Who ordered that?

It might be possible for us to ignore the existence of the negative-energy states, if we only ever consider an isolated electron; we could just declare the positive-energy states to be the ones we are interested in, and ignore the others. However, the problem becomes hard to dismiss once we let the electron interact with another system, such as the electromagnetic field. Under such circumstances, the availability of negative-energy states extending down to $E \to -\infty$ would destabilize the positive-energy electron states, since the electron can repeatedly hop to states with ever more negative energies by shedding energy (e.g., by emitting photons). This is obviously problematic. However, let us wait for a while (till Section II D) to discuss how the stability problem might be resolved.

For now, let us take a closer look at the meaning of the Dirac wavefunction. Its four components represent a four-fold "internal" degree of freedom, distinct from the electron's

ordinary kinematic degrees of freedom. Since there are two energy bands, the assignment of an electron to the upper or lower band (or some superposition thereof) consitutes two degrees of freedom. Each band must then posssess a two-fold degree of freedom (so that $2 \times 2 = 4$), which turns out to be associated with the electron's spin.

To see explicitly how this works, let us pick a representation for the $\hat{\alpha}_\mu$ matrices. The choice of representation determines how the four degrees of freedom are encoded in the individual wavefunction components. We will use the Dirac representation (5.47). In this case, it is convenient to divide the components into upper and lower parts,

$$\psi(\mathbf{r}, t) = \begin{bmatrix} \psi_A(\mathbf{r}, t) \\ \psi_B(\mathbf{r}, t) \end{bmatrix}, \tag{5.49}$$

where ψ_A and ψ_B have two components each. Then, for an eigenstate with energy E and momentum $\mathbf{p}$, applying (5.47) to the Dirac equation (5.41) gives

$$\psi_A = \frac{1}{E - mc^2} \sum_j \hat{\sigma}_j p_j \psi_B, \tag{5.50}$$

$$\psi_B = \frac{1}{E + mc^2} \sum_j \hat{\sigma}_j p_j \psi_A. \tag{5.51}$$

Consider the non-relativistic limit, $|\mathbf{p}| \to 0$, for which E approaches either mc^2 or $-mc^2$. For the upper band ($E \gtrsim mc^2$), the vanishing of the denominator in Eq. (5.50) tells us that the wavefunction is dominated by ψ_A. Conversely, for the lower band ($E \lesssim -mc^2$), Eq. (5.51) tells us that the wavefunction is dominated by ψ_B. We can thus associate the upper (A) and lower (B) components with the band degree of freedom. Note, however, that this is only an approximate association that holds in the non-relativistic limit! In the relativistic regime, upper-band states can have non-vanishing values in the B components, and vice versa. (There does exist a way to make the upper/lower spinor components correspond rigorously to positive/negative energies, but this requires a more complicated representation than the Dirac representation [3].)

C. Dirac electrons in an electromagnetic field

To continue pursuing our objective of interpreting the Dirac wavefunction, we must determine how the electron interacts with an electromagnetic field. We introduce electromagnetism by following the same procedure as in the non-relativistic theory (Section I A): add $-e\Phi(\mathbf{r}, t)$ as a scalar potential function, and add the vector potential via the substitution

$$\hat{\mathbf{p}} \to \hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}}, t). \tag{5.52}$$

Applying this recipe to the Dirac Hamiltonian (5.41) yields

$$i\hbar\,\partial_t \psi = \left\{ \hat{\alpha}_0 mc^2 - e\Phi(\mathbf{r}, t) + \sum_j \hat{\alpha}_j \Big[-i\hbar\,\partial_j + eA_j(\mathbf{r}, t) \Big] c \right\} \psi(\mathbf{r}, t). \tag{5.53}$$

You can check that this has the same gauge symmetry properties as the non-relativistic theory discussed in Section I B.

In the Dirac representation (5.47), Eq. (5.53) reduces to

$$i\hbar\,\partial_t\,\psi_A = \big(+mc^2 - e\Phi\big)\,\psi_A \,+\, \sum_j \hat{\sigma}_j\big(-i\hbar\partial_j + eA_j\big)\,c\,\psi_B \tag{5.54}$$

$$i\hbar\,\partial_t\,\psi_B = \big(-mc^2 - e\Phi\big)\,\psi_B \,+\, \sum_j \hat{\sigma}_j\big(-i\hbar\partial_j + eA_j\big)\,c\,\psi_A, \tag{5.55}$$

where ψ_A and ψ_B are the previously-introduced two-component objects corresponding to the upper and lower halves of the Dirac wavefunction.

In the non-relativistic limit, solutions to the above equations can be cast in the form

$$\begin{aligned}\psi_A(\mathbf{r},t) &= \Psi_A(\mathbf{r},t)\,\exp\left[-i\left(\frac{mc^2}{\hbar}\right)t\right]\\ \psi_B(\mathbf{r},t) &= \Psi_B(\mathbf{r},t)\,\exp\left[-i\left(\frac{mc^2}{\hbar}\right)t\right].\end{aligned} \tag{5.56}$$

The exponentials on the right side are the $\exp(-i\omega t)$ factor corresponding to the rest energy mc^2, which dominates the electron's energy in the non-relativistic limit. (Note that by using $+mc^2$ rather than $-mc^2$, we are explicitly referencing the positive-energy band.) If the electron is in an eigenstate with $\mathbf{p} = 0$ and there are no electromagnetic fields, Ψ_A and Ψ_B would just be constants. Now suppose the electron is non-relativistic but not in a $\mathbf{p} = 0$ eigenstate, and the electromagnetic fields are weak but not necessarily vanishing. In that case, Ψ_A and Ψ_B are functions that vary with t, but slowly.

Plugging this ansatz into Eqs. (5.54)–(5.55) gives

$$i\hbar\,\partial_t\,\Psi_A = -e\Phi\;\Psi_A \,+\, \sum_j \hat{\sigma}_j\big(-i\hbar\partial_j + eA_j\big)c\;\Psi_B \tag{5.57}$$

$$\big(i\hbar\,\partial_t \,+ 2mc^2 + e\Phi\big)\Psi_B = \sum_j \hat{\sigma}_j\big(-i\hbar\partial_j + eA_j\big)c\;\Psi_A. \tag{5.58}$$

On the left side of Eq. (5.58), the $2mc^2$ term dominates over the other two, so

$$\Psi_B \;\approx\; \frac{1}{2mc}\sum_j \hat{\sigma}_j\big(-i\hbar\partial_j + eA_j\big)\;\Psi_A. \tag{5.59}$$

Plugging this into Eq. (5.57) yields

$$i\hbar\,\partial_t\,\Psi_A = \left\{-e\Phi \,+\, \frac{1}{2m}\sum_{jk}\hat{\sigma}_j\hat{\sigma}_k\big(-i\hbar\partial_j + eA_j\big)\big(-i\hbar\partial_k + eA_k\big)\right\}\;\Psi_A. \tag{5.60}$$

Using the identity $\hat{\sigma}_j\hat{\sigma}_k = \delta_{jk}\hat{I} + i\sum_i \varepsilon_{ijk}\sigma_i$:

$$\begin{aligned}i\hbar\,\partial_t\,\Psi_A = \Bigg\{&-e\Phi \,+\, \frac{1}{2m}\big|-i\hbar\nabla + e\mathbf{A}\big|^2\\ &+\, \frac{i}{2m}\sum_{ijk}\varepsilon_{ijk}\hat{\sigma}_i\big(-i\hbar\partial_j + eA_j\big)\big(-i\hbar\partial_k + eA_k\big)\Bigg\}\Psi_A.\end{aligned} \tag{5.61}$$

Look carefully at the last term in the curly brackets. Expanding the square yields

$$\frac{i}{2m}\sum_{ijk}\varepsilon_{ijk}\hat{\sigma}_i\Big(-\partial_j\partial_k - i\hbar e\partial_j A_k - i\hbar e\big[A_k\partial_j + A_j\partial_k\big] + e^2 A_j A_k\Big).$$

Due to the antisymmetry of ε_{ijk}, all terms inside the parentheses that are symmetric under j and k cancel out when summed over. The only survivor is the second term, which gives

$$\frac{\hbar e}{2m}\sum_{ijk}\varepsilon_{ijk}\hat{\sigma}_i\partial_j A_k = \frac{\hbar e}{2m}\hat{\boldsymbol{\sigma}}\cdot\mathbf{B}(\mathbf{r},t), \tag{5.62}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$ is the magnetic field. Hence,

$$i\hbar\,\partial_t\,\Psi_A = \left\{-e\Phi + \frac{1}{2m}\Big|-i\hbar\nabla + e\mathbf{A}\Big|^2 - \left(-\frac{\hbar e}{2m}\,\hat{\boldsymbol{\sigma}}\right)\cdot\mathbf{B}\right\}\Psi_A. \tag{5.63}$$

This is an exact match for Eq. (5.18), except that the Hamiltonian has an additional term of the form $-\hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{B}}$. This additional term corresponds to the potential energy of a magnetic dipole of moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$. The Dirac theory therefore predicts the electron's magnetic dipole moment to be

$$|\boldsymbol{\mu}| = \frac{\hbar e}{2m}. \tag{5.64}$$

Remarkably, this matches the experimentally-observed magnetic dipole moment to about one part in 10^3. The residual mismatch between Eq. (5.64) and the actual magnetic dipole moment of the electron is understood to arise from quantum fluctuations of the electronic and electromagnetic quantum fields. Using the full theory of quantum electrodynamics, that "anomalous magnetic moment" can also be calculated and matches experiment to around one part in 10^9, making it one of the most precise theoretical predictions in physics! For details, see Ref. [2].

It is noteworthy that we did not set out to include spin in the theory, yet it arose, seemingly unavoidably, as a by-product of formulating a relativistic theory of the electron. This is a manifestation of the general principle that relativistic quantum theory is more constrained than non-relativistic quantum theory [1]. Due to the demands imposed by relativistic symmetries, spin is not allowed to be an optional part of the theory of the relativistic electron—it has to be built into the theory at a fundamental level.

D. Positrons and Dirac Field Theory

As noted in Section II B, the stability of the quantum states described by the Dirac equation is threatened by the presence of negative-energy solutions. To get around this problem, Dirac suggested that what we regard as the "vacuum" may actually be a state, called the **Dirac sea**, in which all negative-energy states are occupied. Since electrons are fermions, the Pauli exclusion principle would then forbid decay into the negative-energy states, stabilizing the positive-energy states.

At first blush, the idea seems ridiculous; how can the vacuum contain an infinite number of particles? However, we shall see that the idea becomes more plausible if the Dirac

equation is reinterpreted as a single-particle *construction* which arises from a more fundamental quantum field theory. The Dirac sea idea is an inherently multi-particle concept, and we know from Chapter 4 that quantum field theory is a natural framework for describing multi-particle quantum states. Let us therefore develop this theory.

Consider again the eigenstates of the single-particle Dirac Hamiltonian with definite momenta and energies. Denote the positive-energy wavefunctions by

$$\frac{u_{\mathbf{k}\sigma}\, e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} = \langle \mathbf{r}|\mathbf{k},+,\sigma\rangle, \quad \text{where } \hat{H}|\mathbf{k},+,\sigma\rangle = \epsilon_{\mathbf{k}\sigma}|\mathbf{k},+,\sigma\rangle. \tag{5.65}$$

The negative-energy wavefunctions are

$$\frac{v_{\mathbf{k}\sigma}\, e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} = \langle \mathbf{r}|\mathbf{k},-,\sigma\rangle, \quad \text{where } \hat{H}|\mathbf{k},-,\sigma\rangle = -\epsilon_{\mathbf{k}\sigma}|\mathbf{k},-,\sigma\rangle. \tag{5.66}$$

Note that $|\mathbf{k},-,\sigma\rangle$ denotes a negative-energy eigenstate with momentum $-\hbar\mathbf{k}$, not $\hbar\mathbf{k}$. The reason for this notation, which uses different symbols to label the positive-energy and negative-energy states, will become clear later. Each of the $u_{\mathbf{k}\sigma}$ and $v_{\mathbf{k}\sigma}$ terms are four-component objects (spinors), and for any given $\mathbf{k}$, the set

$$\{u_{\mathbf{k}\sigma}, v_{\mathbf{k},\sigma} \;\mid\; \sigma = 1,2\}$$

forms an orthonormal basis for the four-dimensional spinor space. Thus,

$$\sum_n \left(u^n_{\mathbf{k}\sigma}\right)^* u^n_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}, \;\; \sum_n \left(u^n_{\mathbf{k}\sigma}\right)^* v^n_{\mathbf{k}\sigma'} = 0, \;\; \text{etc.} \tag{5.67}$$

Here we use the notation where $u^n_{\mathbf{k}\sigma}$ is the n-th component of the $u_{\mathbf{k}\sigma}$ spinor, and likewise for the v's.

Following the second quantization procedure from Chapter 4, let us introduce a fermionic Fock space $\mathscr{H}_F$, as well as a set of creation/annihilation operators:

$$\begin{aligned} \hat{b}^\dagger_{\mathbf{k}\sigma} \text{ and } \hat{b}_{\mathbf{k}\sigma} \text{ create/annihilate } |\mathbf{k},+,\sigma\rangle \\ \hat{d}^\dagger_{\mathbf{k}\sigma} \text{ and } \hat{d}_{\mathbf{k}\sigma} \text{ create/annihilate } |\mathbf{k},-,\sigma\rangle. \end{aligned}$$

These obey the fermionic anticommutation relations

$$\begin{aligned} \{\hat{b}_{\mathbf{k}\sigma}, \hat{b}^\dagger_{\mathbf{k}'\sigma'}\} = \delta^3(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma\sigma'}, \quad \{\hat{d}_{\mathbf{k}\sigma}, \hat{d}^\dagger_{\mathbf{k}'\sigma'}\} = \delta^3(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma\sigma'} \\ \{\hat{b}_{\mathbf{k}\sigma}, \hat{b}_{\mathbf{k}'\sigma'}\} = \{\hat{b}_{\mathbf{k}\sigma}, \hat{d}_{\mathbf{k}'\sigma'}\} = \{\hat{d}_{\mathbf{k}\sigma}, \hat{d}_{\mathbf{k}'\sigma'}\} = 0, \;\; \text{etc.} \end{aligned} \tag{5.68}$$

The Hamiltonian is

$$\hat{H} = \int d^3k \sum_\sigma \epsilon_{\mathbf{k}\sigma} \left(\hat{b}^\dagger_{\mathbf{k}\sigma}\hat{b}_{\mathbf{k}\sigma} - \hat{d}^\dagger_{\mathbf{k}\sigma}\hat{d}_{\mathbf{k}\sigma}\right), \tag{5.69}$$

and applying the annihilation operators to the vacuum state $|\varnothing\rangle$ gives zero:

$$\hat{b}_{\mathbf{k}\sigma}|\varnothing\rangle = \hat{d}_{\mathbf{k}\sigma}|\varnothing\rangle = 0. \tag{5.70}$$

When formulating bosonic field theory, we defined a local field annihilation operator that annihilates a particle at a given point $\mathbf{r}$. In the infinite-system limit, this took the form

$$\hat{\psi}(\mathbf{r}) = \int d^3k\ \varphi_{\mathbf{k}}(\mathbf{r})\,\hat{a}_{\mathbf{k}}, \tag{5.71}$$

and the orthonormality of the $\varphi_{\mathbf{k}}$ wavefunctions implied that $[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta^3(\mathbf{r}-\mathbf{r}')$. Similarly, we can use the Dirac Hamiltonian's eigenfunctions (5.65)–(5.66) to define

$$\hat{\psi}_n(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_\sigma \left(u^n_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}}\,\hat{b}_{\mathbf{k}\sigma} + v^n_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}\,\hat{d}_{\mathbf{k}\sigma} \right). \tag{5.72}$$

Note that there are two terms in the parentheses because the positive-energy and negative-energy states are denoted by differently-labeled annihilation operators. Moreover, since the wavefunctions are four-component spinors, the field operators have a spinor index n. Using the spinor orthonormality conditions (5.67) and the anticommutation relations (5.68), we can show that

$$\left\{ \hat{\psi}_n(\mathbf{r}), \hat{\psi}^\dagger_{n'}(\mathbf{r}') \right\} = \delta_{nn'}\,\delta^3(\mathbf{r}-\mathbf{r}'), \tag{5.73}$$

with all other anticommutators vanishing. Hence, $\hat{\psi}_n(\mathbf{r})$ can be regarded as an operator that annihilates a four-component fermion at point $\mathbf{r}$.

Now let us *define* the operators

$$\hat{c}_{\mathbf{k}\sigma} = \hat{d}^\dagger_{\mathbf{k}\sigma}. \tag{5.74}$$

Using these, the fermionic anticommutation relations can be re-written as

$$\begin{aligned} \{\hat{b}_{\mathbf{k}\sigma}, \hat{b}^\dagger_{\mathbf{k}'\sigma'}\} = \delta^3(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma\sigma'}, \quad \{\hat{c}_{\mathbf{k}\sigma}, \hat{c}^\dagger_{\mathbf{k}'\sigma'}\} = \delta^3(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma\sigma'} \\ \{\hat{b}_{\mathbf{k}\sigma}, \hat{b}_{\mathbf{k}'\sigma'}\} = \{\hat{b}_{\mathbf{k}\sigma}, \hat{c}_{\mathbf{k}'\sigma'}\} = \{\hat{c}_{\mathbf{k}\sigma}, \hat{c}_{\mathbf{k}'\sigma'}\} = 0, \quad \text{etc.} \end{aligned} \tag{5.75}$$

Hence $\hat{c}^\dagger_{\mathbf{k}\sigma}$ and $\hat{c}_{\mathbf{k}\sigma}$ formally satisfy the criteria to be regarded as creation and annihilation operators. The particle created by $\hat{c}^\dagger_{\mathbf{k}\sigma}$ is called a **positron**, and is equivalent to the *absence* of a d-type particle (i.e., a negative-energy electron).

The Hamiltonian (5.69) can now be written as

$$\hat{H} = \int d^3k \sum_\sigma \epsilon_{\mathbf{k}\sigma} \left(\hat{b}^\dagger_{\mathbf{k}\sigma}\hat{b}_{\mathbf{k}\sigma} + \hat{c}^\dagger_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma} \right) \ + \ \text{constant}, \tag{5.76}$$

which explicitly shows that the positrons have positive energies (i.e., the absence of a negative-energy particle is equivalent to the presence of a positive-energy particle). With further analysis, which we will skip, it can be shown that the positron created by $\hat{c}^\dagger_{\mathbf{k}\sigma}$ has positive charge e and momentum $\hbar\mathbf{k}$. The latter is thanks to the definition adopted in Eq. (5.66); the absence of a momentum $-\hbar\mathbf{k}$ particle is equivalent to the presence of a momentum $\hbar\mathbf{k}$ particle. As for the field annihilation operator (5.72), it can be written as

$$\hat{\psi}_n(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_\sigma \left(u^n_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}}\,\hat{b}_{\mathbf{k}\sigma} + v^n_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}\,\hat{c}^\dagger_{\mathbf{k}\sigma} \right). \tag{5.77}$$

The c-type annihilation operators do *not* annihilate $|\varnothing\rangle$. However, let us define

$$|\varnothing'\rangle = \prod_{\mathbf{k}\sigma} \hat{d}^\dagger_{\mathbf{k}\sigma}|\varnothing\rangle, \tag{5.78}$$

which is evidently a formal description of the Dirac sea state. Then

$$\hat{c}_{\mathbf{k}\sigma}|\varnothing'\rangle = \hat{d}^\dagger_{\mathbf{k}\sigma}\prod_{\mathbf{k}'\sigma'} \hat{d}^\dagger_{\mathbf{k}'\sigma'}|\varnothing\rangle = 0. \tag{5.79}$$

At the end of the day, we can regard the quantum field theory as being defined in terms of b-type and c-type operators, using the anticommutators (5.75), the Hamiltonian (5.76), and the field operator (5.77), along with the vacuum state $|\varnothing'\rangle$. The elementary particles in this theory are electrons and positrons with strictly positive energies. The single-particle Dirac theory, with its quirky negative-energy states, can then be interpreted as a special construct that maps the quantum field theory into single-particle language. Even though we actually started from the single-particle description, it is the quantum field theory, and its vacuum state $|\varnothing'\rangle$, that is more fundamental.

There are many more details about the Dirac theory that we will not discuss here. One particularly important issue is how the particles transform under Lorentz boosts and other changes in coordinate system. For such details, the reader is referred to Ref. [1].

III. QUANTIZING THE ELECTROMAGNETIC FIELD

Previously (Chapter 4, Sec. IV.C), we have gone through the process of quantizing a scalar boson field. The classical field is decomposed into normal modes, and each mode is quantized by assigning it an independent set of creation and annihilation operators. By comparing the oscillator energies in the classical and quantum regimes, we can derive the Hermitian operator corresponding to the classical field variable, expressed using the creation and annihilation operators. We will use the same approach, with only minor adjustments, to quantize the electromagnetic field.

First, consider a "source-free" electromagnetic field—i.e., with no electric charges and currents. Without sources, Maxwell's equations (in SI units, and in a vacuum) reduce to:

$$\nabla \cdot \mathbf{E} = 0 \tag{5.80}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{5.81}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.82}$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{5.83}$$

Once again, we introduce the scalar potential Φ and vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{5.84}$$
$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5.85}$$

With these relations, Eqs. (5.81) and (5.82) are satisfied automatically via vector identities. The two remaining equations, (5.80) and (5.83), become:

$$\nabla^2\Phi = -\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} \tag{5.86}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \nabla\left[\frac{1}{c^2}\frac{\partial}{\partial t}\Phi + \nabla\cdot\mathbf{A}\right]. \tag{5.87}$$

In the next step, we choose a convenient gauge called the **Coulomb gauge**:

$$\Phi = 0, \quad \nabla\cdot\mathbf{A} = 0. \tag{5.88}$$

(To see that we can always make such a gauge choice, suppose we start out with a scalar potential Φ_0 and vector potential $\mathbf{A}_0$ not satisfying (5.88). Perform a gauge transformation with a gauge field $\Lambda(\mathbf{r},t) = -\int^t dt'\ \Phi_0(\mathbf{r},t')$. The new scalar potential is $\Phi = \Phi_0 + \dot{\Lambda} = 0$; moreover, the new vector potential satisfies

$$\nabla\cdot\mathbf{A} = \nabla\cdot\mathbf{A}_0 - \nabla^2\Lambda = \nabla\cdot\mathbf{A}_0 + \int^t dt'\ \nabla^2\Phi_0(\mathbf{r},t'). \tag{5.89}$$

Upon using Eq. (5.86), we find that $\nabla\cdot\mathbf{A} = 0$.)

In the Coulomb gauge, Eq. (5.86) is automatically satisfied. The sole remaining equation, (5.87), simplifies to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0. \tag{5.90}$$

This has plane-wave solutions of the form

$$\mathbf{A}(\mathbf{r},t) = \left(\mathcal{A}\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{c.c.}\right)\mathbf{e}, \tag{5.91}$$

where $\mathcal{A}$ is a complex number (the **mode amplitude**) that specifies the magnitude and phase of the plane wave, $\mathbf{e}$ is a real unit vector (the **polarization vector**) that specifies which direction the vector potential points along, and "c.c." denotes the complex conjugate of the first term. Referring to Eq. (5.90), the angular frequency ω must satisfy

$$\omega = c|\mathbf{k}|. \tag{5.92}$$

Moreover, since $\nabla\cdot\mathbf{A} = 0$, it must be the case that

$$\mathbf{k}\cdot\mathbf{e} = 0. \tag{5.93}$$

In other words, the polarization vector is perpendicular to the propagation direction. For any given $\mathbf{k}$, we can choose (arbitrarily) two orthogonal polarization vectors.

Now suppose we put the electromagnetic field in a box of volume $V = L^3$, with periodic boundary conditions (we will take $L \to \infty$ at the end). The $\mathbf{k}$ vectors form a discrete set:

$$k_j = \frac{2\pi n_j}{L}, \quad n_j \in \mathbf{Z}, \quad \text{for} \quad j = 1,2,3. \tag{5.94}$$

Then the vector potential field can be decomposed as a superposition of plane waves,

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \left(\mathcal{A}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{c.c.} \right) \mathbf{e}_{\mathbf{k}\lambda}, \quad \text{where} \quad \omega_{\mathbf{k}} = c|\mathbf{k}|. \tag{5.95}$$

Here, λ is a two-fold polarization degree of freedom indexing the two possible orthogonal polarization vectors for each $\mathbf{k}$. (We won't need to specify how exactly these polarization vectors are defined, so long as the definition is used consistently.)

To convert the classical field theory into a quantum field theory, for each $(\mathbf{k},\lambda)$ we define an independent set of creation and annihilation operators:

$$\left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] = \left[\hat{a}^\dagger_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = 0. \tag{5.96}$$

Then the Hamiltonian for the electromagnetic field is

$$\hat{H} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda}, \quad \text{where} \quad \omega_{\mathbf{k}} = c|\mathbf{k}|. \tag{5.97}$$

The vector potential is now promoted into a Hermitian operator in the Heisenberg picture:

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \mathcal{C}_{\mathbf{k}\lambda} \left(\hat{a}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{h.c.} \right) \mathbf{e}_{\mathbf{k}\lambda}. \tag{5.98}$$

Here, $\mathcal{C}_{\mathbf{k}\lambda}$ is a constant to be determined, and "h.c." denotes the Hermitian conjugate. The creation and annihilation operators in this equation are Schrödinger picture ($t = 0$) operators. The particles they create/annihilate are **photons**—elementary particles of light.

To find $\mathcal{C}_{\mathbf{k}\lambda}$, we compare the quantum and classical energies. Suppose the electromagnetic field is in a coherent state $|\alpha\rangle$ such that for any $\mathbf{k}$ and λ,

$$\hat{a}_{\mathbf{k}\lambda}|\alpha\rangle = \alpha_{\mathbf{k}\lambda}|\alpha\rangle \tag{5.99}$$

for some $\alpha_{\mathbf{k}\lambda} \in \mathbb{C}$. From this and Eq. (5.98), we identify the corresponding classical field

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \left(\mathcal{A}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{c.c.} \right) \mathbf{e}_{\mathbf{k}\lambda}, \quad \text{where} \quad \mathcal{C}_{\mathbf{k}\lambda}\alpha_{\mathbf{k}\lambda} = \mathcal{A}_{\mathbf{k}\lambda}. \tag{5.100}$$

For each $\mathbf{k}$ and λ, Eqs. (5.84)–(5.85) give the electric and magnetic fields

$$\mathbf{E}_{\mathbf{k}\lambda} = \left(i\omega_{\mathbf{k}}\mathcal{A}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{c.c.} \right) \mathbf{e}_{\mathbf{k}\lambda} \tag{5.101}$$

$$\mathbf{B}_{\mathbf{k}\lambda} = \left(i\mathcal{A}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{c.c.} \right) \mathbf{k} \times \mathbf{e}_{\mathbf{k}\lambda}. \tag{5.102}$$

In the classical theory of electromagnetism, Poynting's theorem tells us that the total energy carried by a classical plane electromagnetic wave is

$$\begin{aligned} E &= \int_V d^3r\, \frac{\epsilon_0}{2} \left(\left|\mathbf{E}_{\mathbf{k}\lambda}\right|^2 + c^2 \left|\mathbf{B}_{\mathbf{k}\lambda}\right|^2 \right) \\ &= 2\,\epsilon_0\,\omega_{\mathbf{k}}^2\,|\mathcal{A}_{\mathbf{k}\lambda}|^2\,V. \end{aligned} \tag{5.103}$$

Here, V is the volume of the enclosing box, and we have used the fact that terms like $e^{2i\mathbf{k}\cdot\mathbf{r}}$ vanish when integrated over $\mathbf{r}$. Hence, we make the correspondence

$$2\,\epsilon_0\,\omega_{\mathbf{k}}^2\,|\mathcal{C}_{\mathbf{k}\lambda}\alpha_{\mathbf{k}\lambda}|^2\,V = \hbar\omega_{\mathbf{k}}|\alpha_{\mathbf{k}\lambda}|^2 \quad \Rightarrow \quad \mathcal{C}_{\mathbf{k}\lambda} = \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}. \tag{5.104}$$

We thus arrive at the result

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} \\ \hat{\mathbf{A}}(\mathbf{r},t) &= \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\left(\hat{a}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{h.c.}\right)\mathbf{e}_{\mathbf{k}\lambda} \\ \omega_{\mathbf{k}} &= c|\mathbf{k}|, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] = 0. \end{aligned} \tag{5.105}$$

To describe infinite free space rather than a finite-volume box, we take the $L \to \infty$ limit and re-normalize the creation and annihilation operators by the replacement

$$\hat{a}_{\mathbf{k}\lambda} \to \sqrt{\frac{(2\pi)^3}{V}}\,\hat{a}_{\mathbf{k}\lambda}. \tag{5.106}$$

Then the sums over $\mathbf{k}$ become integrals over the infinite three-dimensional space:

$$\begin{aligned} \hat{H} &= \int d^3k \sum_{\lambda} \hbar\omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} \\ \hat{\mathbf{A}}(\mathbf{r},t) &= \int d^3k \sum_{\lambda} \sqrt{\frac{\hbar}{16\pi^3\epsilon_0\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{h.c.}\right)\mathbf{e}_{\mathbf{k}\lambda} \\ \omega_{\mathbf{k}} &= c|\mathbf{k}|, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] = 0. \end{aligned} \tag{5.107}$$

IV. THE ELECTRON-PHOTON INTERACTION

Having derived quantum theories for the electron and the electromagnetic field, we can put them together to describe how electrons interact with the electromagnetic field by absorbing and/or emitting photons. Here, we present the simplest such calculation.

Let $\mathscr{H}_{\text{e}}$ be the Hilbert space for one electron, and $\mathscr{H}_{\text{EM}}$ be the Hilbert space for the electromagnetic field. The combined system is thus described by $\mathscr{H}_e \otimes \mathscr{H}_{\text{EM}}$. We seek a Hamiltonian of the form

$$H = H_e + H_{\text{EM}} + H_{\text{int}}, \tag{5.108}$$

where H_e is the Hamiltonian for the "bare" electron, $H_{\rm EM}$ is the Hamiltonian for the source-free electromagnetic field, and $H_{\rm int}$ is an **interaction Hamiltonian** describing how the electron interacts with photons.

Let us once again adopt the Coulomb gauge, so that the scalar potential is zero, and the electromagnetic field is solely described via the vector potential. In Section I A, we saw that the effect of the vector potential on a charged particle can be described via the substitution

$$\hat{\mathbf{p}} \to \hat{\mathbf{p}} + e\mathbf{A}(\hat{\mathbf{r}}, t). \tag{5.109}$$

In Section II C, we saw that this substitution is applicable not just to non-relativistic particles, but also to fully relativistic particles described by the Dirac Hamiltonian. Previously, we have treated the $\mathbf{A}$ in this substitution as a classical object lacking quantum dynamics of its own. Now, we replace it by the vector potential *operator* derived in Section III:

$$\hat{\mathbf{A}}(\hat{\mathbf{r}}, t) = \begin{cases} \displaystyle\sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_{\mathbf{k}} V}} \left(\hat{a}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} + \text{h.c.} \right) \mathbf{e}_{\mathbf{k}\lambda}, & \text{(finite volume)} \\ \displaystyle\int d^3k \sum_{\lambda} \sqrt{\frac{\hbar}{16\pi^3 \epsilon_0 \omega_{\mathbf{k}}}} \left(\hat{a}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\hat{\mathbf{r}} - \omega_{\mathbf{k}} t)} + \text{h.c.} \right) \mathbf{e}_{\mathbf{k}\lambda}, & \text{(infinite space).} \end{cases} \tag{5.110}$$

Using this, together with either the electronic and electromagnetic Hamiltonians, we can finally describe the photon emission process. Suppose a non-relativistic electron is orbiting an atomic nucleus in an excited state $|1\rangle \in \mathscr{H}_e$. Initially, the photon field is in its vacuum state $|\varnothing\rangle \in \mathscr{H}_{\rm EM}$. Hence, the initial state of the combined system is

$$|\psi_i\rangle = |1\rangle \otimes |\varnothing\rangle. \tag{5.111}$$

Let $H_{\rm int}$ be the Hamiltonian term responsible for photon absorption/emission. If $H_{\rm int} = 0$, then $|\psi_i\rangle$ would be an energy eigenstate. The atom would remain in its excited state forever.

In actuality, $H_{\rm int}$ is not zero, so $|\psi_i\rangle$ is not an energy eigenstate. As the system evolves, the excited electron may decay into its ground state $|0\rangle$ by emitting a photon with energy E, equal to the energy difference between the atom's excited state $|1\rangle$ and ground state $|0\rangle$. For a non-relativistic electron, the Hamiltonian (5.18) yields the interaction Hamiltonian

$$H_{\rm int} = \frac{e}{2m} \left(\hat{\mathbf{p}} \cdot \hat{\mathbf{A}} + \text{h.c.} \right), \tag{5.112}$$

where $\hat{\mathbf{A}}$ must now be treated as a field operator, not a classical field.

Consider the states that $|\psi_i\rangle$ can decay into. There is a continuum of possible final states, each having the form

$$|\psi_f^{(\mathbf{k}\lambda)}\rangle = |0\rangle \otimes \left(\hat{a}^\dagger_{\mathbf{k}\lambda} |\varnothing\rangle \right), \tag{5.113}$$

which describes the electron being in its ground state and the electromagnetic field containing one photon, with wave-vector $\mathbf{k}$ and polarization λ.

According to Fermi's Golden Rule (see Chapter 2), the decay rate is

$$\kappa = \frac{2\pi}{\hbar} \overline{\left| \langle \psi_f^{(\mathbf{k}\lambda)} | \hat{H}_{\rm int} | \psi_i \rangle \right|^2}\, \mathcal{D}(E), \tag{5.114}$$

where $\overline{(\cdots)}$ denotes the average over the possible decay states of energy E (i.e., equal to the energy of the initial state), and $\mathcal{D}(E)$ is the density of states.

To calculate the matrix element $\langle\psi_f^{(\mathbf{k}\lambda)}|\hat{H}_{\rm int}|\psi_i\rangle$, let us use the infinite-volume version of the vector field operator (5.110). (You can check that using the finite-volume version yields the same results; see Exercise 2.) We will use the Schrödinger picture operator, equivalent to setting $t = 0$ in Eq. (5.110). Then

$$\begin{aligned}\langle\psi_f^{(\mathbf{k}\lambda)}|\hat{H}_{\rm int}|\psi_i\rangle = \frac{e}{2m}\int d^3k' \sum_{j\lambda}\sqrt{\frac{\hbar}{16\pi^3\epsilon_0\omega_{\mathbf{k}'}}} \\ \times\Big(\langle 0|\hat{p}_j e^{-i\mathbf{k}\cdot\hat{\mathbf{r}}}|1\rangle + \langle 0|e^{-i\mathbf{k}\cdot\hat{\mathbf{r}}}\hat{p}_j|1\rangle\Big)\, e^j_{\mathbf{k}'\lambda'}\,\langle\varnothing|\hat{a}_{\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}'\lambda'}|\varnothing\rangle.\end{aligned} \tag{5.115}$$

We can now use the fact that $\langle\varnothing|\hat{a}_{\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}'\lambda'}|\varnothing\rangle = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$. Moreover, we approximate the $\exp(-i\mathbf{k}\cdot\hat{\mathbf{r}})$ factors in the brakets with 1; this is a good approximation since the size of a typical atomic orbital ($\lesssim 10^{-9}\,\mathrm{m}$) is much smaller than the optical wavelength ($\sim 10^{-6}\,\mathrm{m}$), meaning that $\exp(-i\mathbf{k}\cdot\mathbf{r})$ does not vary appreciably over the range of positions $\mathbf{r}$ where the orbital wavefunctions are significant. The above equation then simplifies to

$$\langle\psi_f^{(\mathbf{k}\lambda)}|\hat{H}_{\rm int}|\psi_i\rangle \approx \frac{e}{m}\sum_j\sqrt{\frac{\hbar}{16\pi^3\epsilon_0\omega_{\mathbf{k}}}}\;\langle 0|\hat{p}_j|1\rangle\, e^j_{\mathbf{k}\lambda}. \tag{5.116}$$

We can make a further simplification by observing that for $\hat{H}_e = |\hat{\mathbf{p}}|^2/2m + V(\mathbf{r})$,

$$[\hat{H}_e, \hat{\mathbf{r}}] = -i\hbar\mathbf{p}/m \quad\Rightarrow\quad \langle 0|\hat{p}_j|1\rangle = -\frac{imE\mathbf{d}}{\hbar}. \tag{5.117}$$

The complex number $\mathbf{d} = \langle 0|\mathbf{r}|1\rangle$, called the **transition dipole moment**, is easily calculated from the orbital wavefunctions. Thus,

$$\langle\psi_f^{(\mathbf{k}\lambda)}|\hat{H}_{\rm int}|\psi_i\rangle \approx -ie\sqrt{\frac{E}{16\pi^3\epsilon_0}}\,\mathbf{d}\cdot\mathbf{e}_{\mathbf{k}\lambda}. \tag{5.118}$$

$$\left|\langle\psi_f^{(\mathbf{k}\lambda)}|\hat{H}_{\rm int}|\psi_i\rangle\right|^2 \approx \frac{e^2E}{16\pi^3\epsilon_0}\left|\mathbf{d}\cdot\mathbf{e}_{\mathbf{k}\lambda}\right|^2. \tag{5.119}$$

(Check for yourself that Eq. (5.119) should, and does, have units of $[E^2V]$.) We now need the average over the possible photon states $(\mathbf{k},\lambda)$. In taking this average, the polarization vector runs over all possible directions, and a standard angular integration shows that

$$\overline{|\mathbf{d}\cdot\mathbf{e}_{\mathbf{k}\lambda}|^2} = \sum_{j=1}^3 |d_j|^2\,\overline{e_j^2} = \sum_{j=1}^3 |d_j|^2\cdot\frac{1}{3} = \frac{|\mathbf{d}|^2}{3}. \tag{5.120}$$

The last thing we need is the density of photon states. Using the dispersion relation $E = \hbar c|\mathbf{k}|$, we can show (see Exercise 3) that

$$\mathcal{D}(E) = \frac{8\pi E^2}{\hbar^3c^3}. \tag{5.121}$$

This includes a factor of 2 for the photons' two-fold polarization degree of freedom. Putting everything together, we arrive at the following decay rate:

$$\kappa = \frac{e^2\, E^3\, \overline{|\mathbf{d}|^2}}{3\pi\hbar^4 c^3 \epsilon_0} \tag{5.122}$$

We can make this look nicer by defining the dimensionless **fine-structure constant**

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}, \tag{5.123}$$

and defining $\omega = E/\hbar$ as the frequency of the emitted photon. The resulting decay rate is

$$\kappa = \frac{4\alpha\omega^3\, \overline{|\mathbf{d}|^2}}{3c^2}. \tag{5.124}$$

The figure below compares this prediction to experimentally-determined decay rates for the simplest excited states of hydrogen, lithium, and sodium atoms. The experimental data are derived from atomic emission line-widths, and correspond to the rate of spontaneous emission (also called the "Einstein A coefficient") as the excited state decays to the ground state. For the Fermi's Golden Rule curve, we simply approximated the transition dipole moment as $|\mathbf{d}| \approx 10^{-10}\,\mathrm{m}$ (based on the fact that $|\mathbf{d}|$ has units of length, and the length scale of an atomic orbital is about an angstrom); to be more precise, $\mathbf{d}$ ought to be calculated using the actual orbital wavefunctions. Even with the crude approximations we have made, the predictions are within striking distance of the experimental values.

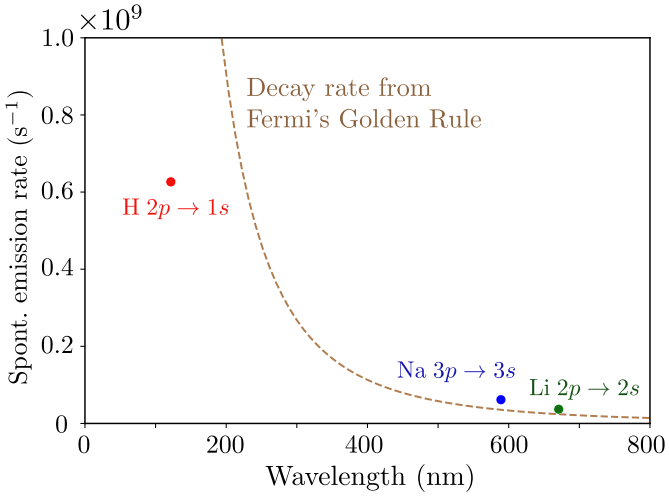


FIG. 1: Spontaneous emission rates (Einstein A coefficients) for the $2p \to 1s$ transition in hydrogen, the $2p \to 2s$ transition in lithium, and the $3p \to 3s$ transition in sodium. Data points extracted from the NIST Atomic Spectra Database (https://www.nist.gov/pml/atomic-spectra-database). The dashed curve shows the decay rate based on Fermi's Golden Rule, with $|\mathbf{d}| \approx 10^{-10}\,\mathrm{m}$.

Exercises

1. In Section III, we derived the vector potential operator, in an infinite volume, to be

$$\hat{\mathbf{A}}(\mathbf{r},t) = \int d^3k \sum_\lambda \sqrt{\frac{\hbar}{16\pi^3\epsilon_0\omega_\mathbf{k}}} \left(\hat{a}_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)} + \text{h.c.}\right) \mathbf{e}_{\mathbf{k}\lambda}. \tag{5.125}$$

   Since $[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}] = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$, the creation and annihilation operators each have units of $[x^{3/2}]$. Prove that $\hat{\mathbf{A}}$ has the same units as the classical vector potential.

2. Repeat the spontaneous decay rate calculation from Section IV using the finite-volume versions of the creation/annihilation operators and the vector potential operator (5.110). Show that it yields the same result (5.123).

3. The density of photon states at energy E is defined as

$$\mathcal{D}(E) = 2\int d^3k\; \delta(E - E_\mathbf{k}), \tag{5.126}$$

   where $E_\mathbf{k} = \hbar c|\mathbf{k}|$. Note the factor of 2 accounting for the polarizations. Prove that

$$\mathcal{D}(E) = \frac{8\pi E^2}{\hbar^3 c^3}, \tag{5.127}$$

   and show that $\mathcal{D}(E)$ has units of $[E^{-1}V^{-1}]$.

Further Reading

[1] F. J. Dyson, *1951 Lectures on Advanced Quantum Mechanics Second Edition*, arxiv:quant-ph/0608140. [link]

[2] A. Zee, *Quantum Field Theory in a Nutshell* (Princeton University Press, 2010).

[3] L. L. Foldy and S. A. Wouthuysen, *On the Dirac Theory of Spin* 1/2 *Particles and Its Non-Relativistic Limit*, Physical Review **78**, 29 (1950). [link]