7. Complex Derivatives

We have previously come across functions that take real inputs and give complex outputs (e.g., solutions to the damped harmonic oscillator that are complex functions of time). For such functions, the derivative with respect to the real input is much like the derivative of a real function of real inputs. It can be calculated by taking the derivatives of the real and imaginary parts separately:

$$\frac{d\psi}{dx} = \frac{d\mathrm{Re}(\psi)}{dx} + i\frac{d\mathrm{Im}(\psi)}{dx}. \tag{7.1}$$

Now consider the more complicated case of a function of a *complex* variable:

$$f(z) \in \mathbb{C}, \quad \text{where} \;\; z \in \mathbb{C}. \tag{7.2}$$

At one level, we could just treat this as a function of two independent real inputs: $f(x, y)$, where $z = x+iy$. However, in doing so we would be disregarding the mathematical structure of the complex input—the fact that z is not merely a collection of two real numbers, but a complex *number* that can participate in algebraic operations. By paying heed to this structure, we will be able to formulate a differential calculus for complex functions.

7.1 Complex continuity and differentiability

The concept of a **continuous complex function** makes use of an "epsilon-delta definition" similar to the definition for functions of real variables (Section 0.7):

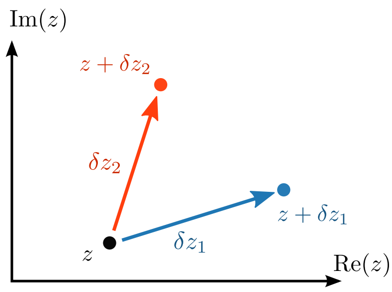
> A complex function $f(z)$ is continuous at $z_0 \in \mathbb{C}$ if, for any $\epsilon > 0$, we can find a $\delta > 0$ such that $\left|z - z_0\right| < \delta \;\Rightarrow\; \left|f(z) - f(z_0)\right| < \epsilon$.

Here, $|\cdots|$ denotes the magnitude of a complex number. If you have difficulty processing this definition, don't worry; it basically says that as z is varied smoothly, there are no abrupt jumps in the value of $f(z)$.

If a function is continuous at a point z, we can define its **complex derivative** as

$$f'(z) \equiv \frac{df}{dz} \equiv \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}. \tag{7.3}$$

This is very similar to the definition of the derivative for a function of a real variable, from Chapter 2. However, there's a complication which doesn't appear in the real case: the infinitesimal δz is complex, yet the above limit expression does not specify its argument (see Section 3.4.1); it simply states the limit as $\delta z \to 0$. The choice of the argument of δz is equivalent to the direction in the complex plane in which δz points, as shown in the following figure:



In principle, we might get different results from the above formula when we plug in different infinitesimals δz and take the limit $\delta z \to 0$, even if $f(z)$ is continuous.

Example—Consider the function $f(z) = z^*$. According to the formula for the complex derivative,

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{z^* + \delta z^* - z^*}{\delta z} = \lim_{\delta z \to 0} \frac{\delta z^*}{\delta z}. \tag{7.4}$$

But if we plug in a real δz, we get a different result than if we plug in an imaginary δz:

$$\begin{aligned} \delta z \in \mathbb{R} \;\; &\Rightarrow \frac{\delta z^*}{\delta z} = 1. \\ \delta z \in i \cdot \mathbb{R} &\Rightarrow \frac{\delta z^*}{\delta z} = -1. \end{aligned} \tag{7.5}$$

To handle this complication, we regard the complex derivative as well-defined *only if* the above definition gives the same answer regardless of the choice of $\arg[\delta z]$. If a function satisfies this property at a point z, we say that it is **complex differentiable** at z.

The preceding example showed that $f(z) = z^*$ is not complex differentiable for any $z \in \mathbb{C}$. The next example shows that $f(z) = z$ is complex differentiable for all $z \in \mathbb{C}$:

Example—The function $f(z) = z$ is complex differentiable for any $z \in \mathbb{C}$, since

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{z + \delta z - z}{\delta z} = \lim_{\delta z \to 0} \frac{\delta z}{\delta z} = 1. \tag{7.6}$$

The result doesn't depend on $\arg[\delta z]$ because the derivative formula simplifies to the fraction $\delta z/\delta z$, which is equal to 1 for any $|\delta z| > 0$. Note that we simplify the fraction to 1 *before* taking the limit $\delta z \to 0$. We can't take the limit first, because $0/0$ is undefined (see Section 1.1.1).

7.2 Analytic functions

If a function $f(z)$ is complex differentiable for all z in some domain $D \subset \mathbb{C}$, then $f(z)$ is said to be **analytic** in D.

The concepts of analyticity and complex differentiability are closely related. It's mainly a matter of terminology: we speak of a function being complex differentiable *at a given point*, and we speak of a function being analytic *in a given domain*.

Example—As shown in the preceding section, $f(z) = z$ is complex-differentiable for any point $z \in \mathbb{C}$. Thence, $f(z) = z$ is analytic in $\mathbb{C}$.

A function's domain of analyticity is often described spatially, in terms of the complex plane. For example, we might say that a function is analytic "everywhere in the complex plane", which means the entire domain $\mathbb{C}$. Or we might say that a function is analytic "in the upper half of the complex plane", meaning for all z such that $\text{Im}(z) > 0$.

7.2.1 Common analytic functions

There is an important class of functions which are analytic over the entire complex plane, or most of the complex plane. These are functions that (i) are expressed in terms of simple algebraic formulas involving z, and (ii) do not contain z^* in the formula.

For example, we have seen that the function $f(z) = z$ is analytic in $\mathbb{C}$. Likewise, $f(z) = \alpha z + \beta$, where α, β are complex constants, is analytic everywhere in $\mathbb{C}$. This can be proven in a similar fashion:

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{[\alpha\,(z + \delta z) + \beta] - [\alpha z + \beta]}{\delta z} && (7.7) \\ &= \lim_{\delta z \to 0} \frac{\alpha \delta z}{\delta z} && (7.8) \\ &= \alpha. && (7.9) \end{aligned}$$

We can also show that $f(z) = z^n$, with $n \in \mathbb{N}$, is analytic everywhere in $\mathbb{C}$:

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{(z + \delta z)^n - z^n}{\delta z} && (7.10) \\ &= \lim_{\delta z \to 0} \frac{(z^n + n z^{n-1} \delta z + \cdots) - z^n}{\delta z} && (7.11) \\ &= n z^{n-1}. && (7.12) \end{aligned}$$

Note that these derivatives have exactly the same algebraic formulas as the corresponding real derivatives. This is no coincidence: to derive the complex derivatives, we take the same series of algebra steps used in deriving the real derivatives.

It should thus be evident that any complex polynomial is analytic everywhere in $\mathbb{C}$. Likewise, functions defined in terms of power series, including the complex exponential and complex sine and cosine, are analytic everywhere in $\mathbb{C}$. Functions involving reciprocals (negative integer powers), such as $f(z) = z^{-1}$ or $f(z) = z^{-2}$, are analytic everywhere *except* at points where $f(z)$ becomes singular (i.e., the denominator goes to zero); we will prove this rigorously in Section 7.3.3.

More generally, whenever a function involves z in some combination of integer polynomials, reciprocals, or functions with power series expansions—and does not involve z^* in an irreducible way—then the function is analytic everywhere except at the singular points. Moreover, the formula for the complex derivative is the same as the corresponding formula for real derivatives.

Example—The function

$$f(z) = \frac{1}{\cos(z)} \qquad (7.13)$$

is analytic everywhere in $\mathbb{C}$, except for values of z such that $\cos(z) = 0$. With a bit of work (try it!), one can show that these z occur at isolated points along the real line, at $z = (m + 1/2)\pi$ where $m \in \mathbb{Z}$, and nowhere else in the complex plane. The complex derivative is

$$f'(z) = \frac{\sin(z)}{[\cos(z)]^2}. \qquad (7.14)$$

The easiest way to prove these statements is to use the Cauchy-Riemann equations, which are discussed below.

One proviso should be kept in mind. For non-integer powers, z^a where $a \notin \mathbb{Z}$, the situation is more complicated because the operation is multi-valued. We'll postpone the discussion of these special operations until the discussion on branch points and branch cuts in Chapter 8.

7.3 The Cauchy-Riemann equations

The **Cauchy-Riemann equations** are a pair of real partial differential equations that provide an alternative way to understand complex derivatives. Their importance comes from the following theorem.

> Suppose f is a complex function that can be written as
>
> $$f(z = x + iy) \;=\; u(x, y) + iv(x, y), \tag{7.15}$$
>
> where $u(x, y)$ and $v(x, y)$ are real functions of two real inputs. Then:
>
> - If f is complex differentiable at a point $z = x + iy$, then u and v satisfy
>
> $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{7.16}$$
>
>   at the point (x, y). These are called the Cauchy-Riemann equations.
>
> - If u and v have continuous first partial derivatives that satisfy the Cauchy-Riemann equations at a point (x, y), then f is complex differentiable at $z = x + iy$.

7.3.1 Proof

We will now prove half of the theorem: the part stating that f being complex differentiable implies the Cauchy-Riemann equations. The converse is left as an exercise.

Suppose f is complex differentiable at some point z. Then there exists a derivative

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}, \tag{7.17}$$

whose value is independent of the argument that we take for the infinitesimal δz. If we take it to be real, i.e. $\delta z = \delta x \in \mathbb{R}$, the expression for the derivative can be written as

$$\begin{aligned}
f'(z) &= \lim_{\delta x \to 0} \frac{f(x + \delta x + iy) - f(x + iy)}{\delta x} && (7.18)\\
&= \lim_{\delta x \to 0} \frac{[u(x + \delta x, y) + iv(x + \delta x, y)] - [u(x, y) + iv(x, y)]}{\delta x} && (7.19)\\
&= \lim_{\delta x \to 0} \frac{[u(x + \delta x, y) - u(x, y)] + i\,[v(x + \delta x, y) - v(x, y)]}{\delta x} && (7.20)\\
&= \left[\lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x}\right] + i\left[\lim_{\delta x \to 0} \frac{v(x + \delta x, y) - v(x, y)}{\delta x}\right] && (7.21)
\end{aligned}$$

On the last line, the quantities in square brackets are the real partial derivatives of u and v with respect to x. Therefore those partial derivatives are well-defined, and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{7.22}$$

On the other hand, we could let the infinitesimal be imaginary, by setting $\delta z = i\delta y$ where

$\delta y \in \mathbb{R}$. Then

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{f(x+iy+i\delta y) - f(x+iy)}{i\delta y} && (7.23) \\ &= \lim_{\delta y \to 0} \frac{[u(x,y+\delta y) + iv(x,y+\delta y)] - [u(x,y) + iv(x,y)]}{i\delta y} && (7.24) \\ &= \lim_{\delta y \to 0} \frac{[u(x,y+\delta y) - u(x,y)] + i\,[v(x,y+\delta y) - v(x,y)]}{i\delta y} && (7.25) \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} && (7.26) \end{aligned}$$

Since $f(z)$ is complex differentiable, these two ways of calculating $f'(z)$ must give the same result:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \qquad (7.27)$$

Since u and v are real functions, we can take the real and imaginary parts of the above equation separately. This yields the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad (7.28)$$

As a corollary, the complex derivative of $f(z)$ can be expressed as:

$$\begin{aligned} \mathrm{Re}\,[f'(z)] &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} && (7.29) \\ \mathrm{Im}\,[f'(z)] &= \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. && (7.30) \end{aligned}$$

7.3.2 Interpretation of the Cauchy-Rieman equations

The basic message of the Cauchy-Riemann equations is that when dealing with analytic functions, the real and imaginary parts cannot be regarded as independent quantities, but are closely intertwined. There are two complementary ways to think about this:

- For an analytic function $f(z)$, the real and imaginary parts of the input z do not independently affect the output value. If I tell you how the function varies in the x direction, by giving you $\partial u/\partial x$ and $\partial v/\partial x$, then you can work out how the function varies in the y direction, by using the Cauchy-Riemann equations to find $\partial u/\partial y$ and $\partial v/\partial y$.
- For the complex outputs of $f(z)$, the real and imaginary parts cannot be regarded as independent. If I tell you how the real part of the output varies, by giving you $\partial u/\partial x$ and $\partial u/\partial y$, then you can work out how the imaginary part of the output varies, by using the Cauchy-Riemann equations to find $\partial v/\partial x$ and $\partial v/\partial y$.

These constraints have profound implications for the mathematical discipline of complex analysis, one of the most important being Cauchy's integral theorem, which we will encounter when studying contour integration in Chapter 9.

7.3.3 Consequences of the Cauchy-Rieman equations

Often, the easiest way to prove that a function is analytic in a given domain is to prove that the Cauchy-Riemann equations are satisfied.

Example—We can use the Cauchy-Riemann equations to prove that the function

$$f(z) = 1/z \tag{7.31}$$

is analytic everywhere, except at $z = 0$. Let us write the function as

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}. \tag{7.32}$$

Hence the real and imaginary component functions are

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = -\frac{y}{x^2+y^2}. \tag{7.33}$$

Except at $x = y = 0$, these functions are differentiable and their partial derivatives satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \tag{7.34}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} = -\frac{\partial u}{\partial y}. \tag{7.35}$$

Moreover, we can use the Cauchy-Riemann equations to prove the following general facts about analytic functions:

- *Compositions of analytic functions are analytic.* If $f(z)$ is analytic in $D \subset \mathbb{C}$ and $g(z)$ is analytic in the range of f, then $g(f(z))$ is analytic in D.
- *Reciprocals of analytic functions are analytic, except at singularities.* If $f(z)$ is analytic in $D \subset \mathbb{C}$, then $1/f(z)$ is analytic everywhere in D except where $f(z) = 0$.

The proofs for these can be obtained via the Cauchy-Riemann equations, and are left as exercises.

7.4 Exercises

1. For each of the following functions $f(z)$, find the real and imaginary component functions $u(x, y)$ and $v(x, y)$, and hence verify whether they satisfy the Cauchy-Riemann equations.

   (a) $f(z) = z$
   (b) $f(z) = z^2$
   (c) $f(z) = |z|$
   (d) $f(z) = |z|^2$
   (e) $f(z) = \exp(z)$
   (f) $f(z) = \cos(z)$
   (g) $f(z) = 1/z$

2. Suppose a function $f(z)$ is well-defined and obeys the Cauchy-Riemann equations at a point z, and the partial derivatives in the Cauchy-Riemann equations are continuous at that point. Show that the function is complex differentiable at that point. Hint: consider an arbitary displacement $\Delta z = \Delta x + i\Delta y$.

3. Prove that products of analytic functions are analytic: if $f(z)$ and $g(z)$ are analytic in $D \subset \mathbb{C}$, then $f(z)g(z)$ is analytic in D. [solution available]

4. Prove that compositions of analytic functions are analytic: if $f(z)$ is analytic in $D \subset \mathbb{C}$ and $g(z)$ is analytic in the range of f, then $g(f(z))$ is analytic in D.

5. Prove that reciprocals of analytic functions are analytic away from poles: if $f(z)$ is analytic in $D \subset \mathbb{C}$, then $1/f(z)$ is analytic everywhere in D except where $f(z) = 0$.

6. Show that if $f(z = x+iy) = u(x,y)+iv(x,y)$ satisfies the Cauchy-Riemann equations, then the real functions u and v each obey Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{7.36}$$

   (Such functions are called "harmonic functions".)

7. We can write the real and imaginary parts of a function in terms of polar coordinates: $f(z) = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. Show that the Cauchy-Riemann equations can be re-written in polar form as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{7.37}$$