4. Complex Numbers

The **imaginary unit**, denoted i, is defined as a solution to the quadratic equation

$$z^2 + 1 = 0. \tag{4.1}$$

In other words, $i = \sqrt{-1}$. As we know, Eq. (4.1) lacks any real number solutions. For this concept to make sense, we must extend our pre-established notions about what numbers are.

Having defined i, we let it take part in the usual arithmetic operations of addition and multiplication, treating it as an algebraic quantity that can participate on the same footing as real numbers. It is one of the most profound discoveries of mathematics that this seemingly arbitrary idea gives rise to powerful computational methods with applications in numerous fields.

4.1 Complex algebra

Any **complex number** z can be written as

$$z = x + iy, \tag{4.2}$$

where x and y are real numbers called the **real part** and the **imaginary part** of z, respectively. The real and imaginary parts are also denoted as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, where Re and Im can be regarded as functions mapping a complex number to a real number.

The set of complex numbers is denoted by $\mathbb{C}$. We can define algebraic operations on complex numbers (addition, subtraction, products, etc.) by following the usual rules of algebra and setting $i^2 = -1$ whenever it shows up.

Example—For $z = x + iy$, where $x, y \in \mathbb{R}$, what are the real and imaginary parts of z^2?

$$z^2 = (x+iy)^2 \tag{4.3}$$
$$= x^2 + 2x(iy) + (iy)^2 \tag{4.4}$$
$$= x^2 - y^2 + 2ixy. \tag{4.5}$$

Hence,

$$\mathrm{Re}(z^2) = x^2 - y^2, \quad \mathrm{Im}(z^2) = 2xy. \tag{4.6}$$

We can also perform power operations on complex numbers, with one caveat: for now, we'll only consider *integer* powers like z^2 or $z^{-1} = 1/z$. Non-integer powers, such as $z^{1/3}$, introduce vexatious complications which are best avoided for now (we'll figure out how to deal with them when studying branch points and branch cuts in Chapter 8).

Another useful fact: real coefficients (and *only* real coefficients) can be freely moved into or out of $\mathrm{Re}(\cdots)$ and $\mathrm{Im}(\cdots)$ operations:

$$\left\{\begin{array}{l}\mathrm{Re}(\alpha z + \beta z') = \alpha\,\mathrm{Re}(z) + \beta\,\mathrm{Re}(z') \\ \mathrm{Im}(\alpha z + \beta z') = \alpha\,\mathrm{Im}(z) + \beta\,\mathrm{Im}(z')\end{array}\right. \quad \text{for } \alpha, \beta \in \mathbb{R}. \tag{4.7}$$

One important consequence is that if we have a complex function of a real variable, $z(t)$, its derivative can be calculated from the derivatives of the real and imaginary parts:

$$\frac{dz}{dt} = \left(\frac{d}{dt}\mathrm{Re}\left[z(t)\right]\right) + i\left(\frac{d}{dt}\mathrm{Im}\left[z(t)\right]\right). \tag{4.8}$$

This can be proven using the definition of the derivative (Chapter 2):

$$\begin{aligned}\mathrm{Re}\left[\frac{dz}{dt}\right] &= \mathrm{Re}\left[\lim_{\delta t\to 0}\frac{z(t+\delta t)-z(t)}{\delta t}\right] &(4.9)\\ &= \lim_{\delta t\to 0}\left[\frac{\mathrm{Re}[z(t+\delta t)]-\mathrm{Re}[z(t)]}{\delta t}\right] &(4.10)\\ &= \frac{d}{dt}\mathrm{Re}\left[z(t)\right]. &(4.11)\end{aligned}$$

The $\mathrm{Im}[\cdots]$ case works out similarly. Note that the infinitesimal quantity δt is real; otherwise, this wouldn't work.

Example—For

$$z(t) = t + it^2, \tag{4.12}$$

the derivative is

$$\frac{dz}{dt} = 1 + 2it. \tag{4.13}$$

4.2 Conjugates and Magnitudes

For each complex number $z = x + iy$, its **complex conjugate** is a complex number whose imaginary part has the sign flipped:

$$z^* \equiv x - iy. \tag{4.14}$$

Conjugation obeys two important properties:

$$\begin{aligned}(z_1 + z_2)^* &= z_1^* + z_2^* &(4.15)\\ (z_1 z_2)^* &= z_1^* z_2^*. &(4.16)\end{aligned}$$

Example—Let us prove that $(z_1 z_2)^* = z_1^* z_2^*$.
First, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$$\begin{aligned}(z_1 z_2)^* &= \left[(x_1 + iy_1)(x_2 + iy_2)\right]^* &(4.17)\\ &= \left[(x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2)\right]^* &(4.18)\\ &= (x_1x_2 - y_1y_2) - i\,(x_1y_2 + y_1x_2) &(4.19)\\ &= (x_1 - iy_1)\,(x_2 - iy_2) &(4.20)\\ &= z_1^* z_2^*. &(4.21)\end{aligned}$$

For a complex number $z = x + iy$, its **magnitude** is

$$|z| \equiv \sqrt{x^2 + y^2}. \tag{4.22}$$

This is a non-negative real number. A complex number and its conjugate have the same magnitude: $|z| = |z^*|$. Also, we can show that complex magnitudes have the property

$$|z_1 z_2| = |z_1|\,|z_2|. \tag{4.23}$$

This property is similar to the "absolute value" operation for real numbers, hence the similar notation.

As a corollary,

$$|z^n| = |z|^n \quad \text{for} \;\; n \in \mathbb{Z}. \tag{4.24}$$

4.3 Euler's formula

Euler's formula is an extremely important result which states that

$$e^{iz} = \cos(z) + i\sin(z). \tag{4.25}$$

To prove this, recall the definition of the exponential from Chapter 1. For real x,

$$\exp(x) \equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \tag{4.26}$$

But such a series would be well-defined even if the input is a complex number, since complex numbers can be added and multiplied by the same rules of algebra as real numbers. This allows us to define the **complex exponential**

$$\exp(z) \equiv e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots \tag{4.27}$$

This is a function that takes complex inputs and gives complex outputs (when the input is real, it gives the same output as the real exponential, a real number). It can be shown to possess all the previously-established algebraic features of the exponential, e.g.,

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2) \quad \text{for} \;\; z_1, z_2 \in \mathbb{C}. \tag{4.28}$$

Likewise, we can define the **complex cosine** and **complex sine** functions using their series formulas (see Section 1.2):

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \tag{4.29}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \tag{4.30}$$

Now, plugging iz into the complex exponential defined in Eq. (4.27) gives

$$\exp(iz) = 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \cdots \tag{4.31}$$

$$= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} + \cdots \tag{4.32}$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right). \tag{4.33}$$

Comparing the two terms in parentheses to Eqs. (4.29)–(4.30), we find that they are perfect matches! Hence, we have proven Eq. (4.25).

One important consequence of Euler's formula is that

$$\left|e^{i\theta}\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1 \qquad \text{for } \theta \in \mathbb{R}. \tag{4.34}$$

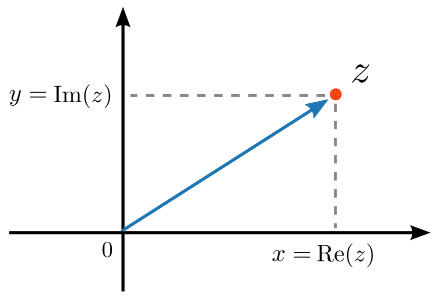
Another consequence is that

$$e^{i\pi} = -1, \tag{4.35}$$

which is a cute little relation between two transcendental constants $e = 2.7182818285\ldots$ and $\pi = 3.141592654\ldots$, by means of the imaginary unit.

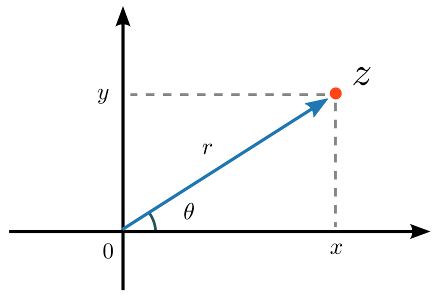
4.4 The complex plane

It is often convenient to regard a complex number as a point on a two-dimensional plane, called the **complex plane**. The real and imaginary parts are the horizontal and vertical Cartesian coordinates in the plane, and the corresponding horizontal (x) and vertical (y) coordinate axes are called the **real axis** and the **imaginary axis**, respectively:



4.4.1 Polar representation

A point in the complex plane can also be represented using polar coordinates. Given $z = x + iy$, we can introduce polar coordinates r and θ (both real numbers):



According to the usual formulas for converting between two-dimensional Cartesian coordinates and polar coordinates,

$$r = \sqrt{x^2 + y^2}, \qquad x = r\cos\theta, \tag{4.36}$$

$$\theta = \tan^{-1}(y/x), \qquad y = r\sin\theta. \tag{4.37}$$

The radial coordinate is equal to the magnitude of the complex number, $|z| = r$ (see Section 4.2). The azimuthal coordinate is called the **argument** of the complex number, and is denoted by $\arg(z) = \theta$.

Note that the complex zero, $z = 0$, has zero magnitude and *undefined* argument.

Using Euler's formula, Eq. (4.25), we can write

$$\begin{aligned} z &= x + iy && (4.38)\\ &= r\cos(\theta) + ir\sin(\theta) && (4.39)\\ &= r\left[\cos(\theta) + i\sin(\theta)\right] && (4.40)\\ &= r\, e^{i\theta}. && (4.41) \end{aligned}$$

Therefore, whenever we can manipulate a complex number into a form Ae^{iB}, where A and B are real, then A is the magnitude and B is the argument. This is used in the following example:

Example—For $z \in \mathbb{C}$, it can be shown that

$$\big|\exp(z)\big| = e^{\mathrm{Re}(z)}, \quad \arg\big[\exp(z)\big] = \mathrm{Im}(z). \tag{4.42}$$

Proof: Let $z = x + iy$, where $x, y \in \mathbb{R}$; then

$$e^z = e^{x+iy} = e^x \, e^{iy}. \tag{4.43}$$

By inspection, the magnitude of this complex number is e^x, and its argument is y.

4.4.2 Geometrical interpretation of complex operations

Using the complex plane, we can give geometric interpretations to the basic operations on complex numbers:

- Addition of two complex numbers can be interpreted as the addition of two coordinate vectors. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

  $$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)\,. \tag{4.44}$$

  Hence, the point corresponding to $z_1 + z_2$ is obtained by adding the two coordinate vectors corresponding to z_1 and z_2. From this, we can geometrically prove a useful inequality relation between complex numbers, called the "triangle inequality":

  $$|z_1 + z_2| \le |z_1| + |z_2|. \tag{4.45}$$

- Complex multiplication can be interpreted as a scaling together with a rotation. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

  $$z_1 z_2 = (r_1 r_2)\, \exp[i(\theta_1 + \theta_2)]. \tag{4.46}$$

  Hence, the point corresponding to $z_1\, z_2$ is obtained by scaling the z_1 coordinate vector by a factor of $|z_2|$, and rotating it by an angle of θ_2 around the origin. In particular, multiplication by $e^{i\theta}$ is equivalent to a rotation by angle θ.

- Complex conjugation (defined in Section 4.2) is equivalent to reflection about the real axis. It moves a point from the upper half of the complex plane to the lower half, or vice versa.

4.4.3 Complex numbers have no ordering

The fact that complex numbers reside in a two-dimensional plane implies that *inequality relations are undefined for complex numbers*. This is a critical difference between complex and real numbers.

Real numbers can be thought of as points on a one-dimensional line (the real line). As a consequence, they can be ordered, meaning that for any two real numbers a and b, one and only one of the following is true:

$$a < b \;\; \text{OR} \;\; a = b \;\; \text{OR} \;\; a > b. \tag{4.47}$$

But since complex numbers lie in a two-dimensional plane, they cannot be compared using "$<$" or "$>$". Given complex numbers z_1 and z_2, it is simply nonsensical to write something like $z_1 < z_2$. (We can, however, write $|z_1| < |z_2|$, since the magnitudes of complex numbers are real numbers.)

4.5 Complex functions

When deriving Euler's formula in Section 4.3, we introduced **complex functions** defined by taking real mathematical functions, like the exponential, and making them accept complex number inputs. Let us take a closer look at these complex functions.

4.5.1 Complex trigonometric functions

As discussed in Section 4.3, the complex sine and cosine functions are defined by the series

$$\left.\begin{aligned}\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots ,\end{aligned}\right. \quad z \in \mathbb{C}. \tag{4.48}$$

It is important to note that the *outputs* of the complex trigonometric functions are complex numbers too.

Some familiar properties of the real trigonometric functions do not apply to the complex versions. For instance, $|\sin(z)|$ and $|\cos(z)|$ are *not* bounded by 1 when z is not real.

We can also write the complex cosine and sine functions in terms of the exponential:

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \tag{4.49}$$

$$\sin(z) = -\frac{i}{2}\left(e^{iz} - e^{-iz}\right). \tag{4.50}$$

This is often a convenient step when solving integrals, as shown in the following example.

Example—Consider the real integral

$$I = \int_0^\infty dx\; e^{-x}\,\cos(x). \tag{4.51}$$

One way to solve this is to use integration by parts, but another way is to use the complex expansion of the cosine function:

$$I = \int_0^\infty dx\; e^{-x}\,\frac{1}{2}\,\left[e^{ix} + e^{-ix}\right] \tag{4.52}$$

$$= \frac{1}{2}\int_0^\infty dx\;\left[e^{(-1+i)x} + e^{(-1-i)x}\right] \tag{4.53}$$

$$= \frac{1}{2}\left[\frac{e^{(-1+i)x}}{-1+i} + \frac{e^{(-1-i)x}}{-1-i}\right]_0^\infty \tag{4.54}$$

$$= -\frac{1}{2}\left(\frac{1}{-1+i} + \frac{1}{-1-i}\right) \tag{4.55}$$

$$= \frac{1}{2}. \tag{4.56}$$

4.5.2 Complex trigonometric identities

Euler's formula provides a convenient way to deal with trigonometric functions. Consider the addition formulas

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \tag{4.57}$$

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2). \tag{4.58}$$

The standard proofs for these formulas are geometric: you draw a figure, and solve a bunch of relations between the angles and sides of the various triangles, making use of the Pythagorean formula. But using the Euler formula, we can prove these algebraically. For example,

$$\cos(z_1)\cos(z_2) = \frac{1}{4}\left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_1}\right) \tag{4.59}$$
$$= \frac{1}{4}\left[e^{i(z_1+z_2)} + e^{i(-z_1+z_2)} + e^{i(z_1-z_2)} + e^{-i(z_1+z_2)}\right] \tag{4.60}$$
$$\sin(z_1)\sin(z_2) = -\frac{1}{4}\left(e^{iz_1} - e^{-iz_1}\right)\left(e^{iz_2} - e^{-iz_1}\right) \tag{4.61}$$
$$= -\frac{1}{4}\left[e^{i(z_1+z_2)} - e^{i(-z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1+z_2)}\right]. \tag{4.62}$$

Thus,

$$\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) = \frac{1}{2}\left[e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right] = \cos(z_1 + z_2). \tag{4.63}$$

As a bonus, these addition formulas now hold for complex inputs as well, not just real inputs.

4.5.3 Hyperbolic functions

Euler's formula also provides us with a link between the trionometric and hyperbolic functions. From the definition of the hyperbolic functions (Section 0.6):

$$\sinh(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right), \quad \cosh(z) = \frac{1}{2}\left(e^{z} + e^{-z}\right). \tag{4.64}$$

Comparing this to Eqs. (4.49) and (4.50), we see that the trigonometric and hyperbolic functions are related by

$$\sin(z) = -i\sinh(iz), \quad \cos(z) = \cosh(iz) \tag{4.65}$$
$$\sinh(z) = -i\sin(iz), \quad \cosh(z) = \cos(iz). \tag{4.66}$$

Using these relations, we can relate the addition formulas for trignometric formulas to the addition formulas for hyperbolic functions, e.g.

$$\cosh(z_1 + z_2) = \cos(iz_1 + iz_2) \tag{4.67}$$
$$= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2) \tag{4.68}$$
$$= \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2). \tag{4.69}$$

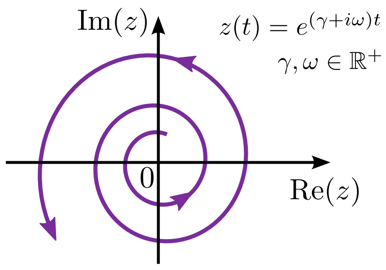
4.6 Trajectories in the complex plane

Suppose we have a function that takes a real input t and outputs a complex number $z(t)$. As t varies, the complex number $z(t)$ forms a curve in the complex plane, which is called the **parametric trajectory** of the function z. Each point on the curve corresponds to the value of $z(t)$ at some t.

Let us study a few examples. First, consider

$$z(t) = e^{i\omega t}, \quad \omega \in \mathbb{R}. \tag{4.70}$$

The trajectory is a circle in the complex plane, centered at the origin and with radius 1. To see why, observe that the function has the form $z(t) = r(t)\, e^{i\theta(t)}$, which has magnitude $r(t) = 1$, and argument $\theta(t) = \omega t$ varying proportionally with t. If ω is positive, the argument increases with t, so the trajectory is counter-clockwise. If ω is negative, the trajectory is clockwise.



On the other hand, consider

$$z(t) = e^{(\gamma + i\omega)t}, \tag{4.71}$$

where $\gamma, \omega \in \mathbb{R}$. For $\gamma = 0$, this reduces to the previous example. For $\gamma \neq 0$, the trajectory is a spiral. To see this, we again observe that this function can be written in the form

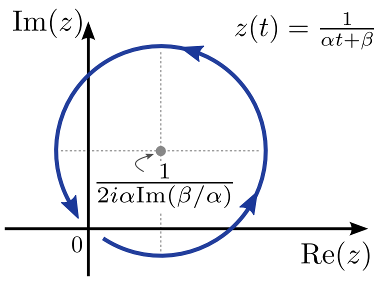
$$z(t) = r(t)\, e^{i\theta(t)}, \tag{4.72}$$

where $r(t) = e^{\gamma t}$ and $\theta = \omega t$. The argument varies proportionally with t, so the trajectory loops around the origin. The magnitude increases with t if γ is positive, and decreases with t if γ is negative. Thus, for instance, if γ and ω are both positive, then the trajectory is an anticlockwise spiral moving outwards from the origin.

Finally, consider

$$z(t) = \frac{1}{t + ib}, \quad b \in \mathbb{R}. \tag{4.73}$$

This trajectory is a circle which passes through the origin, as shown below. The center of the circle is located at $z_0 = -i/(2b)$. Showing this requires a bit of ingenuity, and is left as an exercise. This is an example of something called a Möbius transformation.



4.7 Why complex numbers?

Here is a question that might have occurred to you: if we extend the concept of numbers to complex numbers, why stop here? Why not extend the concept further, and formulate other number systems even more complicated than complex numbers?

As we have seen, complex numbers are appealing mathematical objects because they can be manipulated via the same rules of algebra as real numbers. We can add, subtract, multiply, and divide them without running into any logical inconsistencies. One difference is that complex numbers cannot be ordered, as discussed in Section 4.4.3, but this is not a serious limitation.

Complex numbers are, in a sense, the natural mathematical setting for doing algebra. Arguably, they are even more advantageous than the real numbers for doing algebra because, unlike the real numbers, they are **algebraically closed**, meaning that all complex polynomial equations have solutions in $\mathbb{C}$. The real numbers lack this property: there are real algebraic equations with no solution in $\mathbb{R}$, like $x^2 + 1 = 0$. The algebraic closure of $\mathbb{C}$ is called the Fundamental Theorem of Algebra, which gives an idea of its importance (but we won't delve into the details in this course). One consequence of this is that $\mathbb{C}$ cannot be generalized to a more complicated number system via the same route used to extend $\mathbb{R}$ into $\mathbb{C}$.

However, it is possible to formulate number systems more complicated than the complex numbers, by discarding one or more of the usual rules of algebra. The quaternions are a system of four-component numbers obeying an algebra that is non-commutative (i.e., $ab = ba$ is not generally true). The octonions are a yet more complicated system of eight-component numbers which are not only non-commutative but also non-associative (i.e., $(ab)c = a(bc)$ is not generally true). These and other number systems are occasionally useful in physics and other fields, but overall they are vastly less important than $\mathbb{C}$.

One major reason for the usefulness of complex numbers, compared to quaternions and octonions, is that it's relatively easy to formulate a complex version of calculus. The concepts of derivatives and integrals, which are defined using algebraic limit expressions, can be more-or-less directly applied to complex functions, leading to the subject of **complex analysis**. We shall see later, in Chapter 7, that complex analysis has important implications for *real* calculus; for instance, many real integrals can be easily solved by first generalizing them into complex integrals. By contrast, since quaternions and octonions are not commutative, the very concept of a derivative is tricky to formulate in those number systems.

4.8 Exercises

1. Let $z = x + iy$, where $x, y \in \mathbb{R}$. For each of the following expressions, find (i) the real part, (ii) the imaginary part, (iii) the magnitude, and (iv) the complex argument, in terms of x and y:

   (a) z^2

   (b) $1/z$

   (c) $\exp(z)$

   (d) $\exp(iz)$

   (e) $\cos(z)$

2. Prove that $|z_1 z_2| = |z_1|\,|z_2|$, by using (i) the polar representation, and (ii) the Cartesian representation. [solution available]

3. Prove that $(z_1 z_2)^* = z_1^* z_2^*$, by using (i) the polar representation, and (ii) the Cartesian representation. [solution available]

4. Identify the problem with this chain of equations:
$$-1 = i \cdot i = \sqrt{-1}\,\sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1.$$
[solution available]

5. With the aid of Euler's formula, prove that
$$\cos(3x) = 4[\cos(x)]^3 - 3\cos(x) \tag{4.74}$$
$$\sin(3x) = 3\sin(x) - 4[\sin(x)]^3 \tag{4.75}$$

6. For $z_1, z_2 \in \mathbb{C}$ and $\theta \in \mathbb{R}$, show that $\mathrm{Re}\left[z_1 e^{i\theta} + z_2 e^{-i\theta}\right] = A\cos(\theta) + B\sin(\theta)$, for some $A, B \in \mathbb{R}$. Find explicit expressions for A and B in terms of z_1 and z_2.

7. In Section 4.4, we saw that the conjugation operation corresponds to a reflection about the real axis. What operation corresponds to a reflection about the imaginary axis?

8. Consider the complex function of a real variable $z(t) = 1/(\alpha t + \beta)$, where $\alpha, \beta \in \mathbb{C}$ and $t \in \mathbb{R}$.

   (a) For $\alpha = 1$ and $\beta = i$, show that $z(t)$ can be re-expressed as $z(s) = (1 + e^{is})/(2i)$, where $s \in (-\pi, \pi)$. Hint: find a real mapping $t(s)$.

   (b) Hence, show that the trajectory for arbitrary complex values of α, β has the form of a circle.

9. With the help of a computer plotting program, generate complex trajectories for the following functions (for real inputs $t \in \mathbb{R}$). Explain their key features, including the directions of the trajectories:

   (a) $z(t) = \left[1 + \dfrac{\cos(\beta t)}{2}\right] \exp(it)$, for $\beta = 10$ and for $\beta = \sqrt{5}$.

   (b) $z(t) = -it \pm \sqrt{1 - t^2}$.

   (c) $z(t) = ae^{it} + be^{-it}$, for $a = 1, b = -2$ and for $a = 1, b = 2$.