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GENERALIZED LAGUERRE INTERPOLATION AND PSEUDOSPECTRAL METHOD FOR UNBOUNDED DOMAINS*

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Abstract. In this paper, error estimates for generalized Laguerre–Gauss-type interpolations are derived in nonuniformly weighted Sobolev spaces weighted with $\omega_{\alpha,\beta}(x) = x^{\alpha}e^{-\beta x},\ \alpha > -1, \beta > 0$. Generalized Laguerre pseudospectral methods are analyzed and implemented. Two model problems are considered. The proposed schemes keep spectral accuracy and, with suitable choice of basis functions, lead to sparse and symmetric linear systems.

Key words. generalized Laguerre–Gauss-type interpolations, pseudospectral method, unbounded domains, exterior problems

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1. Introduction.

With the extensive applications of Legendre- and Chebyshev-spectral approximations to PDEs in bounded domains (cf. [2, 3, 4, 6, 7, 8]), considerable progress has been made recently in using spectral methods for solving PDEs in unbounded domains. Among the existing methods, the direct and commonly used approach is based on orthogonal systems in infinite intervals, i.e., the Hermite and Laguerre spectral methods (see, e.g., [5, 6, 9, 10, 17, 19]). In earlier studies, one usually considers Laguerre approximations in spaces weighted with e^{-x}, which are not the most appropriate in some cases. For instance, the approximations of some differential equations in financial mathematics, fluid dynamics, quantum mechanics, and astronomical physics involve different weight functions for derivatives of different orders. In such cases, we have to consider the generalized Laguerre approximation with weight function $\omega_\alpha(x) = x^\alpha e^{-x},\ \alpha > -1$, which was used recently for two-dimensional exterior problems; see [11]. Indeed, from both theoretical and computational points of view, it is more interesting to consider an orthogonal system with a more general weight function: $\omega_{\alpha,\beta}(x) = x^\alpha e^{-\beta x},\ \alpha > -1,\ \beta > 0$. One obvious advantage is that it can provide us a variety of choices of polynomial bases to fit exact solutions of underlying differential equations with various asymptotic behaviors at infinity. Moreover, as we will see later, some other good by-products can be obtained using this new family of orthogonal polynomials.

In actual computations, it is more preferable to use the Laguerre interpolation. As we know, there have been many results on the Laguerre polynomial approximation (e.g., see, [2, 5, 6, 8, 10, 11, 12, 13, 14, 17]), but only a few papers dealing with the error analysis of Laguerre interpolation. Recently, some authors developed the

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Laguerre interpolation—for example, the Laguerre interpolation ($\alpha = 0, \beta = 1$) with its applications to approximation of differential equations (see [19]) and the standard generalized Laguerre interpolation ($\alpha > -1, \beta = 1$), which are very useful for approximation of integral equations (see [15, 16]). The objective of this paper is to analyze the generalized Laguerre–Gauss-type interpolation errors with a more general weight $\omega_{\alpha,\beta}(x), \alpha > -1, \beta > 0$. In the special case of $\alpha = 0, \beta = 1$, our new results are better than the previous ones. Moreover, we derive the approximation results in nonuniformly weighted Sobolev spaces, which enables us to develop and analyze efficient generalized Laguerre pseudospectral approximations of a large class of problems in unbounded domains.

This paper is organized as follows. In section 2, we present some basic results on this new generalized Laguerre–Gauss-type interpolation. In section 3, we establish the main approximation results on the generalized Laguerre–Gauss and Laguerre–Gauss–Radau interpolations, which provide us useful tools for numerical analysis of generalized Laguerre pseudospectral methods for unbounded domains. Section 4 is devoted to the generalized Laguerre pseudospectral method for unbounded domains as an important application of the generalized Laguerre–Gauss interpolation. In section 5, we develop a pseudospectral method for exterior problems as an application of the generalized Laguerre–Gauss–Radau interpolation. In section 6, we present some numerical results, which demonstrate the spectral accuracy of proposed schemes. The final section is for some concluding remarks.

2. Generalized Laguerre–Gauss-type interpolations. In this section, we shall introduce the new generalized Laguerre–Gauss-type interpolations, and study the asymptotic behaviors of the interpolation nodes and weights.

2.1. Notation and preliminaries. Let $\Lambda = (0, \infty)$ and $\chi(x)$ be a certain weight function on Λ in the usual sense. We define the weighted space $L^2_\chi(\Lambda)$ as usual with the inner product $(u, v)_\chi$ and the norm $\|v\|_\chi$. For simplicity, we denote $\partial_x^k v(x) = \frac{d^k}{dx^k} v(x), k \geq 1$. For any integer $m \geq 0$, $H^m_\chi(\Lambda) = \{v \mid \partial_x^k v \in L^2_\chi(\Lambda), 0 \leq k \leq m\}$ with the seminorm $|v|_{m,\chi}$ and the norm $\|v\|_{m,\chi}$. For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi}$ by space interpolation as in [1]. For $\chi(x) \equiv 1$, we drop the subscript χ in the previous notations as usual.

Let $\omega_{\alpha,\beta}(x) = x^\alpha e^{-\beta x}$, $\alpha > -1$, $\beta > 0$. In particular, we denote $\omega_\alpha(x) = \omega_{\alpha,1}(x) = x^\alpha e^{-x}$. The new generalized Laguerre polynomial of degree l is defined by

$$\mathcal{L}_l^{(\alpha,\beta)}(x) = \frac{1}{l!} x^{-\alpha} e^{\beta x} \partial_x^l (x^{l+\alpha} e^{-\beta x}), \quad l = 0, 1, \dots .$$

Let $\mathcal{L}_l^{(\alpha)}(x)$ be the usual generalized Laguerre polynomials that are mutually orthogonal with the weight function $\omega_\alpha(x)$. It is noted that $\mathcal{L}_l^{(\alpha)}(x) = \mathcal{L}_l^{(\alpha,1)}(x)$, and

$$\mathcal{L}_l^{(\alpha,\beta)}(x) = \mathcal{L}_l^{(\alpha)}(y) = \mathcal{L}_l^{(\alpha)}(\beta x), \quad y = \beta x. \tag{2.1}$$

Therefore, it is straightforward to derive the following properties (cf. [18]):

$$\mathcal{L}_l^{(\alpha,\beta)}(0) = \mathcal{L}_l^{(\alpha)}(0) = \frac{\Gamma(l + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(l + 1)}, \quad l \geq 0, \tag{2.2}$$

$$\partial_x \mathcal{L}_l^{(\alpha,\beta)}(x) = -\beta \mathcal{L}_{l-1}^{(\alpha+1,\beta)}(x), \quad l \geq 1, \tag{2.3}$$

$$(l + 1)\mathcal{L}_{l+1}^{(\alpha,\beta)}(x) = (2l + \alpha + 1 - \beta x)\mathcal{L}_l^{(\alpha,\beta)}(x) - (l + \alpha)\mathcal{L}_{l-1}^{(\alpha,\beta)}(x), \quad l \geq 1, \tag{2.4}$$

(2.5)
$$\mathcal{L}_l^{(\alpha,\beta)}(x) = \mathcal{L}_l^{(\alpha+1,\beta)}(x) - \mathcal{L}_{l-1}^{(\alpha+1,\beta)}(x) = \beta^{-1}\Big(\partial_x \mathcal{L}_l^{(\alpha,\beta)}(x) - \partial_x \mathcal{L}_{l+1}^{(\alpha,\beta)}(x)\Big), \quad l \geq 1.$$

The generalized Laguerre polynomials form a complete $L^2_{\omega_{\alpha,\beta}}(\Lambda)$-orthogonal system,

$$\Big(\mathcal{L}_l^{(\alpha,\beta)}, \mathcal{L}_m^{(\alpha,\beta)}\Big)_{\omega_{\alpha,\beta}} = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \quad \gamma_l^{(\alpha,\beta)} = \frac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1}\Gamma(l+1)}, \tag{2.6}$$

where $\delta_{l,m}$ is the Kronecker symbol. Hence, for any $v \in L^2_{\omega_{\alpha,\beta}}(\Lambda)$, we can write

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} \mathcal{L}_l^{(\alpha,\beta)}(x), \qquad \hat{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,\beta)}}(v, \mathcal{L}_l^{(\alpha,\beta)})_{\omega_{\alpha,\beta}}. \tag{2.7}$$

For integer $N > 0$, $\mathbb{P}_N$ stands for the set of algebraic polynomials of degree $\leq N$. We denote by c a generic positive constant independent of N, β, and any function.

2.2. Generalized Laguerre–Gauss and Laguerre–Gauss–Radau interpolations. Let $\xi_{G,N,j}^{(\alpha,\beta)}$ and $\xi_{R,N,j}^{(\alpha,\beta)}$, $0 \leq j \leq N$, be the zeros of $\mathcal{L}_{N+1}^{(\alpha,\beta)}(x)$ and $x\partial_x \mathcal{L}_{N+1}^{(\alpha,\beta)}(x)$, respectively. They are arranged in ascending order. Denote $\omega_{Z,N,j}^{(\alpha,\beta)}$, $0 \leq j \leq N$, $Z = G, R$, the corresponding Christoffel numbers such that

$$\int_\Lambda \phi(x)\omega_{\alpha,\beta}(x)\,dx = \sum_{j=0}^{N} \phi\Big(\xi_{Z,N,j}^{(\alpha,\beta)}\Big)\omega_{Z,N,j}^{(\alpha,\beta)} \quad \forall \phi \in \mathbb{P}_{2N+\lambda_Z}, \tag{2.8}$$

where $\lambda_z = 1$ and 0 for $Z = G$ and R, respectively. In particular, the usual generalized Laguerre–Gauss-type quadrature nodes and weights are denoted by $\xi_{Z,N,j}^{(\alpha)} := \xi_{Z,N,j}^{(\alpha,1)}$ and $\omega_{Z,N,j}^{(\alpha)} := \omega_{Z,N,j}^{(\alpha,1)}$, $Z = G, R$, respectively. Thanks to (2.1), we have $\xi_{Z,N,j}^{(\alpha,\beta)} = \frac{1}{\beta}\xi_{Z,N,j}^{(\alpha)}$. We next derive the expressions of the weights. Indeed,

$$\omega_{G,N,j}^{(\alpha,\beta)} = \frac{1}{\partial_x \mathcal{L}_{N+1}^{(\alpha,\beta)}\big(\xi_{G,N,j}^{(\alpha,\beta)}\big)} \int_\Lambda \frac{\mathcal{L}_{N+1}^{(\alpha,\beta)}(x)}{x - \xi_{G,N,j}^{(\alpha,\beta)}}\omega_{\alpha,\beta}(x)dx, \quad 0 \leq j \leq N, \tag{2.9}$$

which, along with formula (15.3.5) of [18], leads to

(2.10)
$$\omega_{G,N,j}^{(\alpha,\beta)} = \frac{1}{\beta^{\alpha+1}}\omega_{G,N,j}^{(\alpha)} = \frac{\Gamma(N+\alpha+2)}{\beta^\alpha \Gamma(N+2)} \frac{1}{\xi_{G,N,j}^{(\alpha,\beta)}\Big[\partial_x \mathcal{L}_{N+1}^{(\alpha,\beta)}\big(\xi_{G,N,j}^{(\alpha,\beta)}\big)\Big]^2}, \quad 0 \leq j \leq N.$$

Similarly, for the Gauss–Radau weights, we have

(2.11)
$$\omega_{R,N,j}^{(\alpha,\beta)} = \frac{1}{\partial_x\Big[x\partial_x \mathcal{L}_{N+1}^{(\alpha,\beta)}(x)\Big]\big|_{x=\xi_{R,N,j}^{(\alpha,\beta)}}} \int_\Lambda \frac{x\partial_x \mathcal{L}_{N+1}^{(\alpha,\beta)}(x)}{x - \xi_{R,N,j}^{(\alpha,\beta)}}\omega_{\alpha,\beta}(x)dx, \quad 0 \leq j \leq N,$$

which, together with formula (3.6.2) of [6], yields

(2.12)
$$\omega_{R,N,j}^{(\alpha,\beta)} = \frac{1}{\beta^{\alpha+1}}\omega_{R,N,j}^{(\alpha)} = \begin{cases} \dfrac{(\alpha+1)\Gamma^2(\alpha+1)\Gamma(N+1)}{\beta^{\alpha+1}\Gamma(N+\alpha+2)}, & j = 0, \\[2ex] \dfrac{\Gamma(N+\alpha+1)}{\beta^\alpha\Gamma(N+2)} \dfrac{1}{\mathcal{L}_{N+1}^{(\alpha,\beta)}(\xi_{R,N,j}^{(\alpha,\beta)})\partial_x \mathcal{L}_N^{(\alpha,\beta)}(\xi_{R,N,j}^{(\alpha,\beta)})}, & 1 \leq j \leq N. \end{cases}$$

Note that the earlier two types of quadratures have close relations:

$$\xi^{(\alpha,\beta)}_{R,N,j} = \xi^{(\alpha+1,\beta)}_{G,N-1,j-1}, \quad \omega^{(\alpha,\beta)}_{R,N,j} = \left(\xi^{(\alpha,\beta)}_{R,N,j}\right)^{-1} \omega^{(\alpha+1,\beta)}_{G,N-1,j-1}, \quad 1 \le j \le N. \tag{2.13}$$

Indeed, the first identity follows from (2.3). Moreover, using (2.3), (2.9), (2.13), and the definition of $\xi^{(\alpha+1,\beta)}_{G,N-1,j-1}$, we obtain from (2.11) that for $1 \le j \le N$,

$$\begin{aligned}
\omega^{(\alpha,\beta)}_{R,N,j} &= \frac{1}{\partial_x \left(x\mathcal{L}^{(\alpha+1,\beta)}_N(x)\right)\big|_{x=\xi^{(\alpha,\beta)}_{R,N,j}}} \int_\Lambda \frac{x\mathcal{L}^{(\alpha+1,\beta)}_N(x)}{x-\xi^{(\alpha,\beta)}_{R,N,j}} \omega_{\alpha,\beta}(x)\,dx \\
&= \frac{1}{\xi^{(\alpha,\beta)}_{R,N,j}\partial_x \mathcal{L}^{(\alpha+1,\beta)}_N\left(\xi^{(\alpha+1,\beta)}_{G,N-1,j-1}\right)} \int_\Lambda \frac{\mathcal{L}^{(\alpha+1,\beta)}_N(x)}{x-\xi^{(\alpha+1,\beta)}_{G,N-1,j-1}} \omega_{\alpha+1,\beta}(x)\,dx \\
&= \left(\xi^{(\alpha,\beta)}_{R,N,j}\right)^{-1} \omega^{(\alpha+1,\beta)}_{G,N-1,j-1}.
\end{aligned} \tag{2.14}$$

To obtain the interpolation error estimates, it is necessary to study the asymptotic behaviors of generalized Laguerre–Gauss interpolation nodes and weights.

- Using Theorem 8.9.2 of [18], we can verify that for a certain fixed number $\eta > 0$,

$$2\beta^{\frac{1}{2}}\left(\left(\xi^{(\alpha,\beta)}_{G,N,j}\right)\right)^{\frac{1}{2}} = \frac{1}{\sqrt{N+1}}\left(j\pi + \mathcal{O}(1)\right) \quad \text{if } \ 0 < \left(\xi^{(\alpha,\beta)}_{G,N,j}\right) \le \frac{\eta}{\beta}. \tag{2.15}$$

- Theorem 6.31.3 of [18] reveals that for large j,

$$\frac{c_1 j^2}{\beta(N+\frac{\alpha}{2}+\frac{3}{2})} < \left(\xi^{(\alpha,\beta)}_{G,N,j}\right) < \frac{c_2 j^2}{\beta(N+\frac{\alpha}{2}+\frac{3}{2})}, \quad c_1 \cong \frac{\pi^2}{4}, \quad c_2 \cong 4. \tag{2.16}$$

- Let $\tilde{N} = 2(N+1)+\alpha+1$. By Theorem 6.31.2 of [18], the largest node satisfies

$$\xi^{(\alpha,\beta)}_{G,N,N} < \beta^{-1}\left(\tilde{N} + \left(\tilde{N}^2 + \frac{1}{4} - \alpha^2\right)^{1/2}\right) \cong 4\beta^{-1}(N+1). \tag{2.17}$$

- We can verify from formula (15.3.15) of [18] that for a certain fixed number $\eta > 0$,

$$\omega^{(\alpha,\beta)}_{G,N,j} \cong \frac{\pi}{\sqrt{\beta N}} e^{-\beta\xi^{(\alpha,\beta)}_{G,N,j}} \left(\xi^{(\alpha,\beta)}_{G,N,j}\right)^{\alpha+\frac{1}{2}} \quad \text{if } \ 0 < \left(\xi^{(\alpha,\beta)}_{G,N,j}\right) \le \frac{\eta}{\beta}. \tag{2.18}$$

- Let $\xi^{(\alpha,\beta)}_{G,N,-1} := 0$. By the formulae (2.4), (2.5), and (2.7) of [15],

$$\begin{aligned}
\omega^{(\alpha,\beta)}_{G,N,j} &= \frac{1}{\beta^{\alpha+1}}\omega^{(\alpha)}_{G,N,j} \sim \frac{1}{\beta^{\alpha+1}}\omega_\alpha\left(\xi^{(\alpha)}_{G,N,j}\right)\left(\xi^{(\alpha)}_{G,N,j+1} - \xi^{(\alpha)}_{G,N,j}\right) \\
&= \omega_{\alpha,\beta}(\xi^{(\alpha,\beta)}_{G,N,j})\left(\xi^{(\alpha,\beta)}_{G,N,j} - \xi^{(\alpha,\beta)}_{G,N,j-1}\right), \quad 0 \le j \le N.
\end{aligned} \tag{2.19}$$

- Thanks to the relation (2.13), we deduce from (2.18) and (2.19) that

$$\begin{aligned}
\omega^{(\alpha,\beta)}_{R,N,j} &= \left(\xi^{(\alpha+1,\beta)}_{G,N-1,j-1}\right)^{-1} \omega^{(\alpha+1,\beta)}_{G,N-1,j-1} \\
&\cong \frac{\pi}{\sqrt{\beta(N-1)}} e^{-\beta\xi^{(\alpha,\beta)}_{R,N,j}} \left(\xi^{(\alpha,\beta)}_{R,N,j}\right)^{\alpha+\frac{1}{2}} \quad \text{if } \ 0 < \xi^{(\alpha,\beta)}_{R,N,j} \le \frac{\eta}{\beta}, \quad 1 \le j \le N,
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\omega_{R,N,j}^{(\alpha,\beta)} &= \left(\xi_{R,N,j}^{(\alpha,\beta)}\right)^{-1}\omega_{G,N-1,j-1}^{(\alpha+1,\beta)}\\ &\sim \left(\xi_{R,N,j}^{(\alpha,\beta)}\right)^{-1}\omega_{\alpha+1,\beta}\left(\xi_{G,N-1,j-1}^{(\alpha+1,\beta)}\right)\left(\xi_{G,N-1,j-1}^{(\alpha+1,\beta)}-\xi_{G,N-1,j-2}^{(\alpha+1,\beta)}\right)\\ &= \omega_{\alpha,\beta}\left(\xi_{R,N,j}^{(\alpha,\beta)}\right)\left(\xi_{R,N,j}^{(\alpha,\beta)}-\xi_{R,N,j-1}^{(\alpha,\beta)}\right),\quad 1\le j\le N.\end{aligned}\tag{2.21}$$

For notational convenience, we now introduce the discrete inner product and norm,

$$(u,v)_{\omega_{\alpha,\beta},Z,N} = \sum_{j=0}^{N} u\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right)v\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right)\omega_{Z,N,j}^{(\alpha,\beta)},$$

$$\|v\|_{\omega_{\alpha,\beta},Z,N} = (v,v)^{\frac{1}{2}}_{\omega_{\alpha,\beta},Z,N},\quad Z=G,R.$$

By the exactness of (2.8),

$$(\phi,\psi)_{\omega_{\alpha,\beta},Z,N} = (\phi,\psi)_{\omega_{\alpha,\beta}}\quad \forall\phi\psi\in\mathbb{P}_{2N+\delta_Z},\tag{2.22}$$

where $\delta_Z = 1, 0$ for $Z = G, R$, respectively. In particular,

$$\|\phi\|_{\omega_{\alpha,\beta},Z,N} = \|\phi\|_{\omega_{\alpha,\beta}}\quad \forall\phi\in\mathbb{P}_N,\quad Z=G,R.\tag{2.23}$$

The generalized Laguerre–Gauss interpolant $\mathcal{I}_{Z,N,\alpha,\beta}v \in \mathbb{P}_N$ is defined by

$$\mathcal{I}_{Z,N,\alpha,\beta}v\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right) = v\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right),\quad Z=G,R,\quad 0\le j\le N.\tag{2.24}$$

3. Generalized Laguerre interpolation error estimates. In this section, we estimate the interpolation errors in weighted Sobolev spaces, which provide useful tools for the analysis of generalized Laguerre pseudospectral methods.

3.1. $L^2_{\omega_{\alpha,\beta}}(\Lambda)$-orthogonal projection. We first recall the $L^2_{\omega_{\alpha,\beta}}(\Lambda)$-orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega_{\alpha,\beta}}(\Lambda) \to \mathbb{P}_N$, defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega_{\alpha,\beta}} = 0\quad \forall\phi\in\mathbb{P}_N.$$

In order to describe approximation errors precisely, we introduce the nonuniformly weighted Sobolev space $A^r_{\alpha,\beta}(\Lambda)$. For any integer $r \ge 0$, its seminorm and norm are given by

$$|v|_{A^r_{\alpha,\beta}} = \|\partial_x^r v\|_{\omega_{\alpha+r,\beta}},\quad \|v\|_{A^r_{\alpha,\beta}} = \left(\sum_{k=0}^{r}|v|^2_{A^k_{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

For any real $r > 0$, we define the space $A^r_{\alpha,\beta}(\Lambda)$ by space interpolation as in [1].

We have the following basic result; see Theorem 2.1 of [12].

LEMMA 3.1. *For any $v \in A^r_{\alpha,\beta}(\Lambda)$, an integer r, and $0 \le \mu \le r$,*

$$\|P_{N,\alpha,\beta}v - v\|_{A^\mu_{\alpha,\beta}} \le c(\beta N)^{\frac{\mu-r}{2}}|v|_{A^r_{\alpha,\beta}}.\tag{3.1}$$

In the analysis of generalized Laguerre–Gauss–Radau interpolation approximation (cf. the proof of Theorem 3.7), we need to estimate $|P_{N,\alpha,\beta}v(0) - v(0)|$.

LEMMA 3.2. *For any $v \in A^r_{\alpha,\beta}(\Lambda)$ and an integer $r > \alpha + 1$,*

$$|P_{N,\alpha,\beta}v(0) - v(0)| \le c(\beta N)^{\frac{\alpha-r+1}{2}} |v|_{A^r_{\alpha,\beta}}. \tag{3.2}$$

Proof. Let $\lambda_l^{(\beta)} = \beta l$. By virtue of (2.3) and (2.6), we find that for $l \ge r$,

$$|v|^2_{A^r_{\alpha,\beta}} = \sum_{l=r}^{\infty} \beta^{2r} \gamma_{l-r}^{(\alpha+r,\beta)} \big(\hat{v}_l^{(\alpha,\beta)}\big)^2, \quad d_{l,r}^{\alpha,\beta} := \frac{(\lambda_l^{(\beta)})^r \gamma_l^{(\alpha,\beta)}}{\gamma_{l-r}^{(\alpha+r,\beta)}} \le c\beta^{2r}.$$

Therefore,

$$\sum_{l=N+1}^{\infty} \big(\lambda_l^{(\beta)}\big)^r \gamma_l^{(\alpha,\beta)} \big(\hat{v}_l^{(\alpha,\beta)}\big)^2 = \sum_{l=N+1}^{\infty} d_{l,r}^{\alpha,\beta} \gamma_{l-r}^{(\alpha+r,\beta)} \big(\hat{v}_l^{(\alpha,\beta)}\big)^2 \le c|v|^2_{A^r_{\alpha,\beta}}. \tag{3.3}$$

Consequently, using (2.2), (2.6), (3.3), and the Cauchy–Schwarz inequality leads to

$$\begin{aligned}
|P_{N,\alpha,\beta}v(0) - v(0)| &= \Big| \sum_{l=N+1}^{\infty} \hat{v}_l^{(\alpha,\beta)} \mathcal{L}_l^{(\alpha,\beta)}(0) \Big| \\
&\le \left(\sum_{l=N+1}^{\infty} (\lambda_l^{(\beta)})^{-r} (\mathcal{L}_l^{(\alpha,\beta)}(0))^2 (\gamma_l^{(\alpha,\beta)})^{-1} \right)^{\frac{1}{2}} \left(\sum_{l=N+1}^{\infty} (\lambda_l^{(\beta)})^r \gamma_l^{(\alpha,\beta)} (\hat{v}_l^{(\alpha,\beta)})^2 \right)^{\frac{1}{2}} \\
&\le c\beta^{\frac{\alpha-r+1}{2}} \left(\sum_{l=N+1}^{\infty} \frac{\Gamma(l+\alpha+1)}{l^r \Gamma(l+1)} \right)^{\frac{1}{2}} |v|_{A^r_{\alpha,\beta}}.
\end{aligned}$$

By the Stirling formula, $\Gamma(s+1) = \sqrt{2\pi s}\, s^s e^{-s}(1 + \mathcal{O}(s^{-\frac{1}{5}}))$. Thus, for $r > \alpha + 1$,

$$\sum_{l=N+1}^{\infty} \frac{\Gamma(l+\alpha+1)}{l^r \Gamma(l+1)} \le c \sum_{l=N+1}^{\infty} l^{\alpha-r} \le cN^{\alpha-r+1}.$$

This completes the proof. $\square$

The approximation errors stated in Lemma 3.1 are measured in the space $A^\mu_{\alpha,\beta}(\Lambda)$. However, when we apply the generalized Laguerre approximation to numerical solutions of differential and integral equations, we oftentimes need to estimate them in the standard weighted Sobolev space $H^r_{\omega_{\alpha,\beta}}(\Lambda)$, stated later.

LEMMA 3.3. *If $v \in H^\mu_{\omega_{\alpha,\beta}}(\Lambda) \cap A^r_{\alpha-1,\beta}(\Lambda) \cap A^r_{\alpha-\mu,\beta}(\Lambda)$, then for integers $1 \le \mu \le r$,*

$$|P_{N,\alpha,\beta}v - v|_{\mu,\omega_{\alpha,\beta}} \le c\beta^{-\frac{1}{2}} (\beta N)^{\mu - \frac{r}{2}} \big(|v|_{A^r_{\alpha-1,\beta}} + |v|_{A^r_{\alpha-\mu,\beta}} \big). \tag{3.4}$$

Proof. We have

$$|P_{N,\alpha,\beta}v - v|_{1,\omega_{\alpha,\beta}} \le \|P_{N,\alpha,\beta}\partial_x v - \partial_x v\|_{\omega_{\alpha,\beta}} + \|P_{N,\alpha,\beta}\partial_x v - \partial_x P_{N,\alpha,\beta} v\|_{\omega_{\alpha,\beta}}. \tag{3.5}$$

By (3.1) with $\mu = 0$, the first term at the right side of the previous inequality is bounded above by $c(\beta N)^{\frac{1-r}{2}} |v|_{A^r_{\alpha-1,\beta}}$. Hence, it remains to estimate the second term. To do this, let $\partial_x v(x) = \sum_{l=0}^{\infty} \hat{\hat{v}}_l^{(\alpha,\beta)} \mathcal{L}_l^{(\alpha,\beta)}(x)$. By virtue of (2.5) and (2.7), we can derive that $\hat{\hat{v}}_l^{(\alpha,\beta)} = -\beta \sum_{p=l+1}^{\infty} \hat{v}_p^{(\alpha,\beta)}$. Thus, we follow the same lines as in [2, 8] to deduce that

$$
\begin{aligned}
P_{N,\alpha,\beta}\partial_x v(x) - \partial_x P_{N,\alpha,\beta} v(x) &= -\beta \sum_{l=0}^{N} \mathcal{L}_l^{(\alpha,\beta)}(x) \Bigg(\sum_{p=l+1}^{\infty} \hat{v}_p^{(\alpha,\beta)} \Bigg) \\
&\quad + \beta \sum_{l=0}^{N-1} \mathcal{L}_l^{(\alpha,\beta)}(x) \Bigg(\sum_{p=l+1}^{N} \hat{v}_p^{(\alpha,\beta)} \Bigg) = \hat{\hat{v}}_N^{(\alpha,\beta)} \sum_{l=0}^{N} \mathcal{L}_l^{(\alpha,\beta)}(x).
\end{aligned} \tag{3.6}
$$

Accordingly, we use (2.6) and (3.1) with $\mu = 0$ to obtain that

$$
\begin{aligned}
\|P_{N,\alpha,\beta}\partial_x v - \partial_x P_{N,\alpha,\beta} v\|^2_{\omega_{\alpha,\beta}} &= \big(\hat{\hat{v}}_N^{(\alpha,\beta)}\big)^2 \gamma_N^{(\alpha,\beta)} \sum_{l=0}^{N} \gamma_l^{(\alpha,\beta)} \big(\gamma_N^{(\alpha,\beta)}\big)^{-1} \\
&\le \|P_{N-1,\alpha,\beta}\partial_x v - \partial_x v\|^2_{\omega_{\alpha,\beta}} \sum_{l=0}^{N} \gamma_l^{(\alpha,\beta)} \big(\gamma_N^{(\alpha,\beta)}\big)^{-1} \\
&\le c(\beta N)^{1-r} |v|^2_{A^r_{\alpha-1,\beta}} \sum_{l=0}^{N} \gamma_l^{(\alpha,\beta)} \big(\gamma_N^{(\alpha,\beta)}\big)^{-1}.
\end{aligned} \tag{3.7}
$$

If $\alpha \ge 0$, then $\gamma_l^{(\alpha,\beta)}$ increases as l increases. In this case,

$$
\sum_{l=0}^{N} \gamma_l^{(\alpha,\beta)} \big(\gamma_N^{(\alpha,\beta)}\big)^{-1} \le N+1. \tag{3.8}
$$

For $-1 < \alpha < 0$, we use the Stirling formula to deduce that for a suitably large integer $M < N$ and $l \ge M$,

$$
\gamma_l^{(\alpha,\beta)} \sim \beta^{-\alpha-1} \Big(1 + \frac{\alpha}{l}\Big)^{l+\frac{1}{2}} (l+\alpha)^{\alpha} \sim \beta^{-\alpha-1} l^{\alpha}. \tag{3.9}
$$

Hence, for certain $c_1 > 0$,

$$
\sum_{l=0}^{N} \gamma_l^{(\alpha,\beta)} \big(\gamma_N^{(\alpha,\beta)}\big)^{-1} \le cN^{-\alpha} \Bigg(c_1 + c \sum_{l=M}^{N} l^{\alpha} \Bigg) \le cN^{-\alpha}(c_1 + cN^{1+\alpha}) \le cN. \tag{3.10}
$$

Inserting (3.8) and (3.10) into (3.7), we obtain the desired result with $\mu = 1$.

Now, we use induction to derive the desired result with $\mu \ge 2$. We shall use the following inverse inequality:

$$
\|\phi\|_{r,\omega_{\alpha,\beta}} \le c(\beta N)^r \|\phi\|_{\omega_{\alpha,\beta}} \quad \forall \phi \in \mathbb{P}_N, \quad r > 0.
$$

Assume that (3.4) holds for $\mu - 1$. Then we obtain that

$$
\begin{aligned}
&|P_{N,\alpha,\beta} v - v|_{\mu,\omega_{\alpha,\beta}} \le |P_{N,\alpha,\beta}\partial_x v - \partial_x v|_{\mu-1,\omega_{\alpha,\beta}} + |P_{N,\alpha,\beta}\partial_x v - \partial_x P_{N,\alpha,\beta} v|_{\mu-1,\omega_{\alpha,\beta}} \\
&\quad \le c\beta^{-\frac{1}{2}} (\beta N)^{\mu-1-\frac{r-1}{2}} (|v|_{A^r_{\alpha-2,\beta}} + |v|_{A^r_{\alpha-\mu,\beta}}) + c(\beta N)^{\mu-1} \|P_{N,\alpha,\beta}\partial_x v \\
&\qquad - \partial_x P_{N,\alpha,\beta} v\|_{\omega_{\alpha,\beta}} \le c\beta^{-\frac{1}{2}} (\beta N)^{\mu-\frac{r}{2}-\frac{1}{2}} (|v|_{A^r_{\alpha-2,\beta}} + |v|_{A^r_{\alpha-\mu,\beta}}) \\
&\qquad + c\beta^{-\frac{1}{2}} (\beta N)^{\mu-\frac{r}{2}} |v|_{A^r_{\alpha-1,\beta}}.
\end{aligned} \tag{3.11}
$$

By the definition of $|\cdot|_{A^r_{\alpha,\beta}}$, we have that

$$|v|_{A^r_{\alpha-2,\beta}} = \|\partial_x^r v\|_{\omega_{\alpha+r-2,\beta}} \le c(|v|_{A^r_{\alpha-\mu,\beta}} + |v|_{A^r_{\alpha-1,\beta}}).$$

This fact with (3.11) implies the desired result. ∎

3.2. Generalized Laguerre interpolation approximations. We first study the stability of generalized Laguerre–Gauss interpolation.

THEOREM 3.4. *For any* $v \in H^1_{\omega_{\alpha,\beta}}(\Lambda) \cap A^1_{\alpha,\beta}(\Lambda)$,

$$\|\mathcal{I}_{G,N,\alpha,\beta} v\|_{\omega_{\alpha,\beta}} \le c\Big(\beta^{-1} N^{-\frac{1}{2}} |v|_{1,\omega_{\alpha,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}} \|v\|_{A^1_{\alpha,\beta}}\Big). \tag{3.12}$$

Proof. By (2.23) and (2.24),

$$\|\mathcal{I}_{G,N,\alpha,\beta} v\|^2_{\omega_{\alpha,\beta}} = \|\mathcal{I}_{G,N,\alpha,\beta} v\|^2_{\omega_{\alpha,\beta},G,N} = \sum_{j=0}^{N} v^2\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)\omega^{(\alpha,\beta)}_{G,N,j} := A_N + B_N, \tag{3.13}$$

where

$$A_N = \sum_{\xi^{(\alpha,\beta)}_{G,N,j} \le \frac{\eta}{\beta}} v^2\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)\omega^{(\alpha,\beta)}_{G,N,j}, \quad B_N = \sum_{\xi^{(\alpha,\beta)}_{G,N,j} > \frac{\eta}{\beta}} v^2\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)\omega^{(\alpha,\beta)}_{G,N,j}.$$

We first estimate A_N. For simplicity of statements, let

$$\Delta^{(\alpha,\beta)}_j = \Big[\xi^{(\alpha,\beta)}_{G,N,j-1},\ \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)\Big], \quad |\Delta^{(\alpha,\beta)}_j| = \xi^{(\alpha,\beta)}_{G,N,j} - \xi^{(\alpha,\beta)}_{G,N,j-1},$$
$$\delta^{(\alpha,\beta)}_{j,+} = \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} + \big(\xi^{(\alpha,\beta)}_{G,N,j-1}\big)^{\frac{1}{2}}, \quad \delta^{(\alpha,\beta)}_{j,-} = \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} - \big(\xi^{(\alpha,\beta)}_{G,N,j-1}\big)^{\frac{1}{2}}.$$

By (13.7) of [2], we know that for any $u \in H^1(a,b)$,

$$\sup_{x\in[a,b]} |u(x)| \le c\bigg(\frac{1}{\sqrt{b-a}}\|u\|_{L^2(a,b)} + \sqrt{b-a}\|\partial_x u\|_{L^2(a,b)}\bigg). \tag{3.14}$$

Thus, by (2.18) and (3.14),

$$\begin{aligned}
A_N &\le \frac{c}{\sqrt{\beta N}} \sum_{\xi^{(\alpha,\beta)}_{G,N,j} \le \frac{\eta}{\beta}} \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} \sup_{x\in\Delta^{(\alpha,\beta)}_j} |x^\alpha v^2(x)| \\
&\le \frac{c}{\sqrt{\beta N}} \sum_{\xi^{(\alpha,\beta)}_{G,N,j} \le \frac{\eta}{\beta}} \bigg(\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} \big(\delta^{(\alpha,\beta)}_{j,+}\big)^{-1}\big(\delta^{(\alpha,\beta)}_{j,-}\big)^{-1} \|x^{\frac{\alpha}{2}} v\|^2_{L^2\left(\Delta^{(\alpha,\beta)}_j\right)} \\
&\quad + \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} \delta^{(\alpha,\beta)}_{j,+}\delta^{(\alpha,\beta)}_{j,-}\Big(\|x^{\frac{\alpha}{2}}\partial_x v\|^2_{L^2\left(\Delta^{(\alpha,\beta)}_j\right)} + \|x^{\frac{\alpha}{2}-1} v\|^2_{L^2\left(\Delta^{(\alpha,\beta)}_j\right)}\Big)\bigg).
\end{aligned} \tag{3.15}$$

We now bound the terms in the previous summation. Using (2.15) yields

(3.16)
$$\begin{aligned}
\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} \delta^{(\alpha,\beta)}_{j,+}\delta^{(\alpha,\beta)}_{j,-} \|x^{\frac{\alpha}{2}-1} v\|^2_{L^2\left(\Delta^{(\alpha,\beta)}_j\right)} &\le \big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{1}{2}} \big(\delta^{(\alpha,\beta)}_{j,+}\big)^2 \int_{\Delta^{(\alpha,\beta)}_j} x^{\alpha-2} v^2(x)dx \\
&\le c\big(\xi^{(\alpha,\beta)}_{G,N,j}\big)^{\frac{3}{2}} \big(\xi^{(\alpha,\beta)}_{G,N,j-1}\big)^{-2} \int_{\Delta^{(\alpha,\beta)}_j} x^{\alpha} v^2(x)dx \\
&\le c\sqrt{\beta N}\|x^{\frac{\alpha}{2}} v\|^2_{L^2\left(\Delta^{(\alpha,\beta)}_j\right)}.
\end{aligned}$$

The expression (2.15) implies that for $0 < \xi_{G,N,j}^{(\alpha,\beta)} \leq \frac{\eta}{\beta}$,

$$\delta_{j,-}^{(\alpha,\beta)} \sim \frac{1}{\sqrt{\beta N}}, \quad \big(\xi_{G,N,j}^{(\alpha,\beta)}\big)^{\frac{1}{2}}\big(\delta_{j,+}^{(\alpha,\beta)}\big)^{-1} \leq c, \quad \big(\xi_{G,N,j}^{(\alpha,\beta)}\big)^{\frac{1}{2}}\delta_{j,+}^{(\alpha,\beta)} \leq \frac{c}{\beta}. \tag{3.17}$$

Hence, plugging (3.16) and (3.17) into (3.15) gives

$$\begin{aligned} A_N &\leq c \sum_{\Delta_j^{(\alpha,\beta)}} \Big(\|x^{\frac{\alpha}{2}} v\|^2_{L^2\left(\Delta_j^{(\alpha,\beta)}\right)} + \beta^{-2} N^{-1} \|x^{\frac{\alpha}{2}} \partial_x v\|^2_{L^2\left(\Delta_j^{(\alpha,\beta)}\right)} \Big) \\ &\leq c(\|v\|^2_{\omega_{\alpha,\beta}} + \beta^{-2} N^{-1} |v|^2_{1,\omega_{\alpha,\beta}}). \end{aligned} \tag{3.18}$$

We next estimate B_N in (3.13). By (2.19) and (2.17),

$$\begin{aligned} B_N &\leq c \sum_{\xi_{G,N,j}^{(\alpha,\beta)} > \frac{\eta}{\beta}} v^2\big(\xi_{G,N,j}^{(\alpha,\beta)}\big)\omega_{\alpha,\beta}\big(\xi_{G,N,j}^{(\alpha,\beta)}\big)\Big(\xi_{G,N,j}^{(\alpha,\beta)} - \xi_{G,N,j-1}^{(\alpha,\beta)}\Big) \\ &\leq c \sup_{x > \frac{\eta}{\beta}} |v^2(x)\omega_{\alpha+1,\beta}(x)| \sum_{\xi_{G,N,j}^{(\alpha,\beta)} > \frac{\eta}{\beta}} \frac{1}{\xi_{G,N,j}^{(\alpha,\beta)}}\Big(\xi_{G,N,j}^{(\alpha,\beta)} - \xi_{G,N,j-1}^{(\alpha,\beta)}\Big) \\ &\leq c \sup_{x > \frac{\eta}{\beta}} |v^2(x)\omega_{\alpha+1,\beta}(x)| \int_{\frac{\eta}{\beta}}^{4\beta^{-1}(N+1)} \frac{1}{x} dx. \end{aligned}$$

By a similar argument as in the derivation of Lemma 2.2 of [11], we deduce that

$$\sup_{x\in\Lambda} |v^2(x)\omega_{\alpha+1,\beta}(x)| \leq \max(\alpha+1, 2/\beta)\|v\|^2_{A^1_{\alpha,\beta}}. \tag{3.19}$$

Consequently,

$$B_N \leq c(1 + 1/\beta)\ln N \|v\|^2_{A^1_{\alpha,\beta}}. \tag{3.20}$$

The combination of (3.13), (3.18), (3.20), and the fact $\|v\|_{\omega_{\alpha,\beta}} \leq \|v\|_{A^1_{\alpha,\beta}}$ leads to the desired result. □

With the aid of the previous theorem, we are able to estimate the interpolation error.

THEOREM 3.5. *If $v \in A^r_{\alpha-1,\beta}(\Lambda) \cap A^r_{\alpha,\beta}(\Lambda)$, then for integer $r \geq 1$,*

$$\|\mathcal{I}_{G,N,\alpha,\beta} v - v\|_{\omega_{\alpha,\beta}} \leq c(\beta N)^{\frac{1}{2}-\frac{r}{2}} \Big(\beta^{-1}|v|_{A^r_{\alpha-1,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{A^r_{\alpha,\beta}}\Big). \tag{3.21}$$

If, in addition, $v \in A^r_{\alpha-\mu,\beta}(\Lambda)$, then for integers $1 \leq \mu \leq r$,

$$\begin{aligned} |\mathcal{I}_{G,N,\alpha,\beta} v - v|_{\mu,\omega_{\alpha,\beta}} \leq c(\beta N)^{\mu+\frac{1}{2}-\frac{r}{2}} \Big(&\beta^{-1}(|v|_{A^r_{\alpha-1,\beta}} + N^{-\frac{1}{2}}|v|_{A^r_{\alpha-\mu,\beta}}) \\ &+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{A^r_{\alpha,\beta}}\Big). \end{aligned} \tag{3.22}$$

Proof. The use of (3.12), (3.1), and (3.4) with $\mu = 1$ leads to

$$\begin{aligned} &\|\mathcal{I}_{G,N,\alpha,\beta} v - P_{N,\alpha,\beta} v\|_{\omega_{\alpha,\beta}} = \|\mathcal{I}_{G,N,\alpha,\beta}(P_{N,\alpha,\beta} v - v)\|_{\omega_{\alpha,\beta}} \\ &\quad \leq c\beta^{-1}N^{-\frac{1}{2}}|P_{N,\alpha,\beta} v - v|_{1,\omega_{\alpha,\beta}} + c(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|P_{N,\alpha,\beta} v - v\|_{A^1_{\alpha,\beta}} \\ &\quad \leq c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\Big(\beta^{-1}|v|_{A^r_{\alpha-1,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{A^r_{\alpha,\beta}}\Big). \end{aligned} \tag{3.23}$$

Thus, using the previous formula and (3.1) with $\mu = 0$ yields

$$\begin{aligned} \|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta}} &\le \|\mathcal{I}_{G,N,\alpha,\beta}v - P_{N,\alpha,\beta}v\|_{\omega_{\alpha,\beta}} + \|P_{N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta}} \\ &\le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\Big(\beta^{-1}|v|_{A^r_{\alpha-1,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{A^r_{\alpha,\beta}}\Big). \end{aligned} \tag{3.24}$$

This implies (3.21). Next, by (3.4), (3.23), and the inverse inequality as before, we deduce that

$$\begin{aligned} |\mathcal{I}_{G,N,\alpha,\beta}v - v|_{\mu,\omega_{\alpha,\beta}} &\le |\mathcal{I}_{G,N,\alpha,\beta}v - P_{N,\alpha,\beta}v|_{\mu,\omega_{\alpha,\beta}} + |P_{N,\alpha,\beta}v - v|_{\mu,\omega_{\alpha,\beta}} \\ &\le c(\beta N)^{\mu}\|\mathcal{I}_{G,N,\alpha,\beta}v - P_{N,\alpha,\beta}v\|_{\omega_{\alpha,\beta}} + c\beta^{-\frac{1}{2}}(\beta N)^{\mu-\frac{r}{2}}(|v|_{A^r_{\alpha-1,\beta}} + |v|_{A^r_{\alpha-\mu,\beta}}) \\ &\le c(\beta N)^{\mu+\frac{1}{2}-\frac{r}{2}}\Big(\beta^{-1}\big(|v|_{A^r_{\alpha-1,\beta}} + N^{-\frac{1}{2}}|v|_{A^r_{\alpha-\mu,\beta}}\big) + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{A^r_{\alpha,\beta}}\Big). \end{aligned} \tag{3.25}$$

This completes the proof. ∎

We now turn to the generalized Laguerre–Gauss–Radau interpolation. We first study the stability of interpolation, stated later.

THEOREM 3.6. *For any* $v \in H^1_{\omega_{\alpha,\beta}}(\Lambda) \cap A^1_{\alpha,\beta}(\Lambda)$,

$$\begin{aligned} \|\mathcal{I}_{R,N,\alpha,\beta}v\|_{\omega_{\alpha,\beta}} \le c\Big(&(\beta N)^{-\frac{\alpha+1}{2}}|v(0)| + \beta^{-1}N^{-\frac{1}{2}}|v|_{1,\omega_{\alpha,\beta}} \\ &+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|v\|_{A^1_{\alpha,\beta}}\Big). \end{aligned} \tag{3.26}$$

In particular, for $|\alpha| < 1$,

$$\|\mathcal{I}_{R,N,\alpha,\beta}v\|_{\omega_{\alpha,\beta}} \le c\Big(\beta^{-1}N^{-\frac{1}{2}}|v|_{1,\omega_{\alpha,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|v\|_{A^1_{\alpha,\beta}}\Big). \tag{3.27}$$

Proof. Let η be the positive constant in (2.15). By the exactness (2.23),

$$\|\mathcal{I}_{R,N,\alpha,\beta}v\|^2_{\omega_{\alpha,\beta}} = \|\mathcal{I}_{R,N,\alpha,\beta}v\|^2_{\omega_{\alpha,\beta},R,N} = v^2(0)\omega^{(\alpha,\beta)}_{R,N,0} + \widetilde{A}_N + \widetilde{B}_N,$$

where

$$\widetilde{A}_N = \sum_{0<\xi^{(\alpha,\beta)}_{R,N,j}\le\frac{\eta}{\beta}} v^2(\xi^{(\alpha,\beta)}_{R,N,j})\omega^{(\alpha,\beta)}_{R,N,j}, \quad \widetilde{B}_N = \sum_{\xi^{(\alpha,\beta)}_{R,N,j}>\frac{\eta}{\beta}} v^2(\xi^{(\alpha,\beta)}_{R,N,j})\omega^{(\alpha,\beta)}_{R,N,j}.$$

Using the Stirling formula, we have $\omega^{(\alpha,\beta)}_{R,N,0} \le c(\beta N)^{-\alpha-1}$. On the other hand, we observe from (2.13) that the interior nodes $\xi^{(\alpha,\beta)}_{R,N,j}$, $1 \le j \le N$, satisfy asymptotic properties (2.15) and (2.16), while the corresponding weights $\omega^{(\alpha,\beta)}_{R,N,j}$, $1 \le j \le N$, fulfill (2.20) and (2.21). Thus, we can follow the same lines as in the proof of Theorem 3.4 to derive that

$$\widetilde{A}_N \le c(\|v\|^2_{\omega_{\alpha,\beta}} + \beta^{-2}N^{-1}|v|^2_{1,\omega_{\alpha,\beta}}), \quad \widetilde{B}_N \le c(1+\beta^{-1})\ln N\|v\|^2_{A^1_{\alpha,\beta}}.$$

Then the result (3.26) follows from the previous statements.

We next prove (3.27). For any $x \in [0, \frac{1}{\beta}]$ and $|\alpha| < 1$,

$$|v(x) - v(0)| \le \left(\int_0^{\frac{1}{\beta}} x^{-\alpha}e^{\beta x}\,dx\right)^{\frac{1}{2}}\|\partial_x v\|_{L^2_{\omega_{\alpha,\beta}}(0,\frac{1}{\beta})} \le c\beta^{\frac{\alpha-1}{2}}\|\partial_x v\|_{L^2_{\omega_{\alpha,\beta}}(0,\frac{1}{\beta})}. \tag{3.28}$$

Now, let $|v(x_*)| = \min_{x\in[0,1/\beta]} |v(x)|$. Clearly, for $|\alpha| < 1$,

$$|v(x_*)| \le \beta \int_0^{\frac{1}{\beta}} |v(x)|\, dx \le c\beta^{\frac{\alpha+1}{2}} \|v\|_{L^2_{\omega_{\alpha,\beta}}(0,\frac{1}{\beta})}. \tag{3.29}$$

The previous formula with (3.28) gives

$$|v(0)| \le |v(x_*)| + |v(x_*) - v(0)| \le c\Big(\beta^{\frac{\alpha+1}{2}} \|v\|_{\omega_{\alpha,\beta}} + \beta^{\frac{\alpha-1}{2}} |v|_{1,\omega_{\alpha,\beta}}\Big). \tag{3.30}$$

If $0 \le \alpha < 1$, then by (3.30) and the fact $\|v\|_{\omega_{\alpha,\beta}} \le \|v\|_{A^1_{\alpha,\beta}}$, we derive that

$$(\beta N)^{-\frac{\alpha+1}{2}} |v(0)| \le c\Big(\beta^{-1} N^{-\frac{1}{2}} |v|_{1,\omega_{\alpha,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}} \|v\|_{A^1_{\alpha,\beta}}\Big),$$

which, along with (3.26), leads to (3.27) with $0 \le \alpha < 1$. For $-1 < \alpha < 0$, we change slightly the derivation of (3.28) to obtain that for any $x \in [0, \frac{1}{\beta}]$,

$$|v(x) - v(0)| \le \int_0^{\frac{1}{\beta}} |\partial_x v(x)|\, dx \le c\beta^{\frac{\alpha}{2}} \|\partial_x v\|_{L^2_{\omega_{\alpha+1,\beta}}(0,\frac{1}{\beta})}.$$

Correspondingly, (3.30) becomes

$$|v(0)| \le c\Big(\beta^{\frac{\alpha+1}{2}} \|v\|_{\omega_{\alpha,\beta}} + \beta^{\frac{\alpha}{2}} \|\partial_x v\|_{\omega_{\alpha+1,\beta}}\Big) \le c\beta^{\frac{\alpha+1}{2}} (1+\beta^{-\frac{1}{2}}) \|v\|_{A^1_{\alpha,\beta}}.$$

Then the result (3.27) with $-1 < \alpha < 0$ follows from formula (3.26) and the fact $N^{-\frac{\alpha+1}{2}} \le c$. ☐

The following two theorems describe the error of interpolation $\mathcal{I}_{R,N,\alpha,\beta} v$.

THEOREM 3.7. *If $v \in A^r_{\alpha,\beta}(\Lambda) \cap A^r_{\alpha-1,\beta}(\Lambda)$, then for an integer $r \ge 1$ and $r > \alpha + 1$,*

$$\|\mathcal{I}_{R,N,\alpha,\beta} v - v\|_{\omega_{\alpha,\beta}} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}} \Big(\beta^{-1} |v|_{A^r_{\alpha-1,\beta}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}} |v|_{A^r_{\alpha,\beta}}\Big). \tag{3.31}$$

In particular, if $|\alpha| < 1$, then the previous result holds for all integers $r \ge 1$.

Proof. As a consequence of (3.26),

$$\begin{aligned}\|\mathcal{I}_{R,N,\alpha,\beta} v - P_{N,\alpha,\beta} v\|_{\omega_{\alpha,\beta}} &= \|\mathcal{I}_{R,N,\alpha,\beta}(P_{N,\alpha,\beta} v - v)\|_{\omega_{\alpha,\beta}} \\ &\le c(\beta N)^{-\frac{\alpha+1}{2}} |P_{N,\alpha,\beta} v(0) - v(0)| + c\beta^{-1} N^{-\frac{1}{2}} |P_{N,\alpha,\beta} v - v|_{1,\omega_{\alpha,\beta}} \\ &\quad + c(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}} \|P_{N,\alpha,\beta} v - v\|_{A^1_{\alpha,\beta}}.\end{aligned} \tag{3.32}$$

According to Lemma 3.2, the first term on the right-hand side of (3.32) is bounded above by $c(\beta N)^{-\frac{r}{2}} |v|_{A^r_{\alpha,\beta}}$ for an integer $r > \alpha + 1$. The other two terms can be estimated by using Lemmas 3.1 and 3.3 with $\mu = 1$ (cf. the proof of (3.21)).

If $|\alpha| < 1$, we use (3.27) to derive (3.32), which does not contain the term $|P_{N,\alpha,\beta} v(0) - v(0)|$, and consequently does not require $r > \alpha + 1$. ☐

We can follow the same approach as for the proof of (3.22) to derive the following result.

THEOREM 3.8. *If $v \in A^r_{\alpha,\beta}(\Lambda) \cap A^r_{\alpha-1,\beta}(\Lambda) \cap A^{r-1}_{\alpha-\mu,\beta}(\Lambda)$, then for integers $1 \le \mu \le r$ and $r > \alpha + 1$,*

$$\begin{aligned}|\mathcal{I}_{R,N,\alpha,\beta} v - v|_{\mu,\omega_{\alpha,\beta}} \le\, & c(\beta N)^{\mu+\frac{1}{2}-\frac{r}{2}} \Big(\beta^{-1}\Big(|v|_{A^r_{\alpha-1,\beta}} + N^{-\frac{1}{2}} |v|_{A^r_{\alpha-\mu,\beta}}\Big) \\ & + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}} |v|_{A^r_{\alpha,\beta}}\Big).\end{aligned} \tag{3.33}$$

In particular, for $|\alpha| < 1$, the previous result holds for all integers $r \ge 1$.

4. Generalized Laguerre pseudospectral method for unbounded domains. This section is devoted to the generalized pseudospectral method based on the generalized Laguerre–Gauss interpolation. Throughout this section, let $\Omega = \Lambda \times S$ and S be the unit spherical surface, $S = \{(\lambda, \theta) | \ 0 \le \lambda < 2\pi, \ -\frac{\pi}{2} \le \theta < \frac{\pi}{2}\}$. The Laplacian operator on Ω is given by

$$\Delta v(\rho,\lambda,\theta) = \frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho v(\rho,\lambda,\theta)) + \frac{1}{\rho^2\cos\theta}\partial_\theta(\cos\theta\partial_\theta v(\rho,\lambda,\theta)) + \frac{1}{\rho^2\cos^2\theta}\partial_\lambda^2 v(\rho,\lambda,\theta).$$

We consider the following problem:

$$\begin{cases} -\Delta W(\rho,\lambda,\theta) + \mu W(\rho,\lambda,\theta) = F(\rho,\lambda,\theta), \quad \mu > 0, \quad \text{in } \Omega, \\ W(\rho,\lambda+2\pi,\theta) = W(\rho,\lambda,\theta). \end{cases} \tag{4.1}$$

Here, we look for the solution of (4.1) such that $\rho^{\frac{1}{2}} W(\rho,\lambda,\theta) \to 0$ as $\rho \to 0$ and $\rho^{\frac{3}{2}} W(\rho,\lambda,\theta) \to 0$ as $\rho \to \infty$. In addition, the solution $W(\rho,\lambda,\theta)$ satisfies the pole condition, namely, $\partial_\lambda W(\rho,\lambda,\theta) = 0$ for $\theta = \pm\frac{\pi}{2}$.

It is noted that the usual weighted (with the weight $e^{-\beta\rho}$) Galerkin variational formulation of (4.1), on which the generalized Laguerre approximations are often based, is not well posed. One possible way to remedy this deficiency is to find a suitable variable transform such that the weighted variational formulation of the transformed equation becomes well posed. Motivated by [10], we make the variable transform

$$W(\rho,\lambda,\theta) = e^{-\frac{\beta}{2}\rho} U(\rho,\lambda,\theta), \quad F(\rho,\lambda,\theta) = e^{-\frac{\beta}{2}\rho} f(\rho,\lambda,\theta), \tag{4.2}$$

which converts (4.1) into

$$\begin{aligned} &-\partial_\rho^2 U(\rho,\lambda,\theta) - \frac{1}{\rho}(2-\beta\rho)\partial_\rho U(\rho,\lambda,\theta) - \frac{1}{\rho^2\cos\theta}\partial_\theta(\cos\theta\partial_\theta U(\rho,\lambda,\theta)) \\ &\quad - \frac{1}{\rho^2\cos^2\theta}\partial_\lambda^2 U(\rho,\lambda,\theta) + \frac{1}{\rho}\Big(\mu\rho + \beta - \frac{\beta^2}{4}\rho\Big) U(\rho,\lambda,\theta) = f(\rho,\lambda,\theta). \end{aligned} \tag{4.3}$$

To focus on our main idea, we consider only the spherically symmetric case, in which U and f are independent of λ and θ, denoted by $U(\rho)$ and $f(\rho)$, respectively. Accordingly,

$$-\partial_\rho^2 U(\rho) - \frac{1}{\rho}(2-\beta\rho)\partial_\rho U(\rho) + \frac{1}{\rho}\Big(\mu\rho + \beta - \frac{\beta^2}{4}\rho\Big) U(\rho) = f(\rho). \tag{4.4}$$

In addition, $\rho^{\frac{1}{2}} U(\rho) \to 0$ as $\rho \to 0$ and $\rho^{\frac{3}{2}} e^{-\frac{\beta}{2}\rho} U(\rho)$ as $\rho \to \infty$.

With the previous general setup, we now derive a weak formulation of (4.4). First, we observe that for any $v \in H^1_{\omega_{2,\beta}}(\Lambda)$, we have $\partial_\rho v(\rho) = o(\rho^{-\frac{3}{2}})$ and $v(\rho) = o(\rho^{-\frac{1}{2}})$ as $\rho \to 0$, and $\partial_\rho v(\rho) \sim v(\rho) = o(\rho^{-\frac{3}{2}} e^{\frac{\beta\rho}{2}})$ as $\rho \to \infty$. Consequently, if $v \in H^1_{\omega_{2,\beta}}(\Lambda) \cap L^2_{\omega_{1,\beta}}(\Lambda)$, then $\rho^2 v(\rho)\partial_\rho v(\rho) e^{-\beta\rho} \to 0$ as $\rho \to 0, \infty$. Hence, we obtain a weak formulation of (4.4). It is to find $U \in H^1_{\omega_{2,\beta}}(\Lambda) \cap L^2_{\omega_{1,\beta}}(\Lambda)$ such that

$$a_{\mu,\beta}(U,v) = (f,v)_{\omega_{2,\beta}} \quad \forall v \in H^1_{\omega_{2,\beta}}(\Lambda) \cap L^2_{\omega_{1,\beta}}(\Lambda), \tag{4.5}$$

where the bilinear form is defined by

$$a_{\mu,\beta}(u,v) = (\partial_\rho u, \partial_\rho v)_{\omega_{2,\beta}} + \Big(\mu - \frac{\beta^2}{4}\Big)(u,v)_{\omega_{2,\beta}} + \beta(u,v)_{\omega_{1,\beta}}.$$

One can verify that $a_{\mu,\beta}(\cdot,\cdot)$ is continuous and elliptic in $\big(H^1_{\omega_{2,\beta}}(\Lambda)\cap L^2_{\omega_{1,\beta}}(\Lambda)\big)^2$. Indeed,

$$|a_{\mu,\beta}(u,v)| \le c\big((1+\beta)\|u\|_{1,\omega_{2,\beta}}+\beta^{\frac12}\|u\|_{\omega_{1,\beta}}\big)\big((1+\beta)\|v\|_{1,\omega_{2,\beta}}+\beta^{\frac12}\|v\|_{\omega_{1,\beta}}\big), \tag{4.6}$$

and for $\mu > \frac{\beta^2}{4}$, we have

$$a_{\mu,\beta}(v,v) \ge c\big(\|v\|^2_{1,\omega_{2,\beta}}+\beta\|v\|^2_{\omega_{1,\beta}}\big). \tag{4.7}$$

Therefore, if $f \in (H^1_{\omega_{2,\beta}}(\Lambda)\cap L^2_{\omega_{1,\beta}}(\Lambda))'$, then (4.5) admits a unique solution.

The corresponding pseudospectral scheme for (4.5) is to seek $u_N(\rho)\in\mathbb{P}_N$ such that

$$a_{\mu,\beta,N}(u_N,\phi) = (f,\phi)_{\omega_{2,\beta},G,N} \quad \forall\phi\in\mathbb{P}_N, \tag{4.8}$$

where

$$a_{\mu,\beta,N}(u,v) = (\partial_\rho u,\partial_\rho v)_{\omega_{2,\beta},G,N} + \Big(\mu-\frac{\beta^2}{4}\Big)(u,v)_{\omega_{2,\beta},G,N} + \beta(u,v)_{\omega_{1,\beta},G,N}.$$

According to (2.22), (4.8) is equivalent to

$$a_{\mu,\beta}(u_N,\phi) = (\mathcal{I}_{G,N,2,\beta}f,\phi)_{\omega_{2,\beta}} \quad \forall\phi\in\mathbb{P}_N. \tag{4.9}$$

Before analyzing the convergence of (4.8), we first consider a special orthogonal projection $P^1_{N,\beta}: H^1_{\omega_{2,\beta}}(\Lambda)\cap L^2_{\omega_{1,\beta}}(\Lambda)\to\mathbb{P}_N$, defined by

$$\big(\partial_\rho(P^1_{N,\beta}v-v),\partial_\rho\phi\big)_{\omega_{2,\beta}} + \big(P^1_{N,\beta}v-v,\phi\big)_{\omega_{2,\beta}} + \big(P^1_{N,\beta}v-v,\phi\big)_{\omega_{1,\beta}} = 0 \quad \forall\phi\in\mathbb{P}_N. \tag{4.10}$$

To analyze its approximation error, we need the following two imbedding inequalities which are the special cases of Lemmas 2.1 and 2.2 of [12].

- If $v\in L^2_{\omega_{0,\beta}}(\Lambda)$, $\partial_\rho v\in L^2_{\omega_{2,\beta}}(\Lambda)$, and $v(\frac{1}{\beta})=0$, then

$$\|v\|_{\omega_{0,\beta}} \le c\|\partial_\rho v\|_{\omega_{2,\beta}}. \tag{4.11}$$

- If $v\in H^1_{\omega_{2,\beta}}(\Lambda)\cap L^2_{\omega_{0,\beta}}(\Lambda)$, then

$$\|v\|^2_{\omega_{2,\beta}} \le 8\beta^{-2}\big(\|\partial_\rho v\|^2_{\omega_{2,\beta}}+\|v\|^2_{\omega_{0,\beta}}\big). \tag{4.12}$$

LEMMA 4.1. *For any* $v\in H^1_{\omega_{2,\beta}}(\Lambda)\cap L^2_{\omega_{1,\beta}}(\Lambda)\cap A^r_{1,\beta}(\Lambda)$ *and integer* $r\ge1$,

$$\|P^1_{N,\beta}v-v\|_{1,\omega_{2,\beta}} + \|P^1_{N,\beta}v-v\|_{\omega_{1,\beta}} \le c(1+\beta^{-1})(\beta N)^{\frac{1-r}{2}}|v|_{A^r_{1,\beta}}. \tag{4.13}$$

Proof. By projection theorem and the Cauchy–Schwarz inequality,

$$\begin{aligned}\|P^1_{N,\beta}v-v\|^2_{1,\omega_{2,\beta}} + \|P^1_{N,\beta}v-v\|^2_{\omega_{1,\beta}} &\le \|\phi-v\|^2_{1,\omega_{2,\beta}} + \|\phi-v\|^2_{\omega_{1,\beta}}\\ &\le |\phi-v|^2_{1,\omega_{2,\beta}} + \frac32\|\phi-v\|^2_{\omega_{2,\beta}} + \frac12\|\phi-v\|^2_{\omega_{0,\beta}} \quad \forall\phi\in\mathbb{P}_N.\end{aligned}$$

Taking $\phi(\rho) = P_{N,1,\beta}v(\rho) - P_{N,1,\beta}v(\frac1\beta) + v(\frac1\beta)$, we have from (4.11), (4.12), and (3.1) that

$$\begin{aligned}\|P^1_{N,\beta}v-v\|^2_{1,\omega_{2,\beta}} + \|P^1_{N,\beta}v-v\|^2_{\omega_{1,\beta}} &\le c(1+\beta^{-2})\|\partial_\rho(\phi-v)\|^2_{\omega_{2,\beta}}\\ &= c(1+\beta^{-2})|P_{N,1,\beta}v-v|^2_{A^1_{1,\beta}} \le c(1+\beta^{-2})(\beta N)^{1-r}|v|^2_{A^r_{1,\beta}}. \qquad \square\end{aligned}$$

We now go back to the convergence analysis of scheme (4.8). Let $U_N = P^1_{N,\beta}U$; then by (4.5) and (4.10),

$$a_{\mu,\beta}(U_N,\phi) = -G(\phi) + (\mathcal{I}_{G,N,2,\beta}f,\phi)_{\omega_{2,\beta}} \quad \forall\phi\in\mathbb{P}_N, \tag{4.14}$$

where

$$G(\phi) = \Big(\mu - \frac{\beta^2}{4} - 1\Big)(U-U_N,\phi)_{\omega_{2,\beta}} + (\beta-1)(U-U_N,\phi)_{\omega_{1,\beta}} + (\mathcal{I}_{G,N,2,\beta}f - f,\phi)_{\omega_{2,\beta}}.$$

Set $\widetilde{U}_N = u_N - U_N$. Then by (4.9) and (4.14),

$$a_{\mu,\beta}(\widetilde{U}_N,\phi) = G(\phi) \quad \forall\phi\in\mathbb{P}_N. \tag{4.15}$$

Taking $\phi = \widetilde{U}_N$ in the previous formula and using (4.7) give

$$\|\widetilde{U}_N\|^2_{1,\omega_{2,\beta}} + \beta\|\widetilde{U}_N\|^2_{\omega_{1,\beta}} \le c|G(\widetilde{U}_N)|. \tag{4.16}$$

Hence, it suffices to estimate $|G(\widetilde{U}_N)|$. For simplicity, we shall use the following notation:

$$\begin{aligned}
B^{(1)}_{N,\beta,r}(v) &= c(1+\beta^2)^2(1+\beta^{-1})^2(\beta N)^{1-r}|v|^2_{A^r_{1,\beta}},\\
B^{(2)}_{N,\beta,r}(v) &= c(1+\beta)^2(1+\beta^{-1})^2(\beta N)^{1-r}|v|^2_{A^r_{1,\beta}},\\
B^{(3)}_{N,\beta,s}(v) &= c(\beta N)^{1-s}\Big(\beta^{-2}|v|^2_{A^{s-1}_{1,\beta}} + (1+\beta^{-1})\ln N|v|^2_{A^s_{2,\beta}}\Big).
\end{aligned}$$

By virtue of (4.13) and (3.21), for integers $r,s\ge 1$,

$$|G(\widetilde{U}_N)| \le B^{(1)}_{N,\beta,r}(U) + B^{(2)}_{N,\beta,r}(U) + B^{(3)}_{N,\beta,s}(f) + \frac{1}{2}\|\widetilde{U}_N\|^2_{\omega_{2,\beta}} + \frac{\beta}{2}\|\widetilde{U}_N\|^2_{\omega_{1,\beta}}.$$

Plugging the previous formula into (4.16) leads to an estimate for $\|\widetilde{U}_N\|^2_{1,\omega_{2,\beta}} + \beta\|\widetilde{U}_N\|^2_{\omega_{1,\beta}}$. Since $U - u_N = U - P^1_{N,\beta}U - \widetilde{U}_N$, we use (4.13) again to reach the following conclusion.

THEOREM 4.2. *Let U and u_N be the solutions of* (4.5) *and* (4.8), *respectively, and let $\mu > \frac{1}{4}\beta^2$. If $U\in A^r_{1,\beta}(\Lambda)$ and $f\in A^s_{1,\beta}(\Lambda)\cap A^s_{2,\beta}(\Lambda)$ with integers $r,s\ge 1$, then*

$$\|U-u_N\|^2_{1,\omega_{2,\beta}} + \beta\|U-u_N\|^2_{\omega_{1,\beta}} \le c\Big(B^{(1)}_{N,\beta,r}(U) + B^{(2)}_{N,\beta,r}(U) + B^{(3)}_{N,\beta,s}(f)\Big).$$

REMARK 4.1. *After solving $u_N(\rho)$ from* (4.8), *we evaluate the numerical solution of the original problem by $w_N(\rho) = e^{-\frac{\beta}{2}\rho}u_N(\rho)$. Indeed, a direct computation leads to*

$$\begin{aligned}
\|W-w_N\|_{1,\hat\omega_2} + \sqrt{\beta}\|W-w_N\|_{\hat\omega_1} &\le (1+\beta)\|U-u_N\|_{1,\omega_{2,\beta}} + \sqrt{\beta}\|U-u_N\|_{\omega_{1,\beta}}\\
&= \mathcal{O}(N^{\frac{1-r}{2}} + (\ln N)^{\frac{1}{2}}N^{\frac{1-s}{2}}),
\end{aligned}$$

where $\hat\omega_\alpha(\rho) = \rho^\alpha = \omega_{\alpha,0}(\rho)$. A combination of the previous formula and (3.19) *with $\alpha = 1$ yields*

$$\begin{aligned}
\sup_{\rho\in\Lambda}|\rho(W-w_N)| &= \sup_{\rho\in\Lambda}|(U-u_N)\rho e^{-\frac{\beta}{2}\rho}| \le c\|U-u_N\|_{A^1_{1,\beta}}\\
&\le c(|U-u_N|_{1,\omega_{2,\beta}} + \|U-u_N\|_{\omega_{1,\beta}}) = \mathcal{O}(N^{\frac{1-r}{2}} + (\ln N)^{\frac{1}{2}}N^{\frac{1-s}{2}}).
\end{aligned}$$

Hence, a spectral accuracy is expected from theoretical analysis.

REMARK 4.2. *Given $\mu > 0$, we can always choose the adjustable factor β such that $\mu > \frac{1}{4}\beta^2$, which guarantees the well-posedness of our Galerkin formulation.*

5. Generalized Laguerre pseudospectral method for exterior problems. This section is for the generalized Laguerre pseudospectral method based on Gauss–Radau interpolation for exterior problems. As an example, we consider the following equation induced by the spherically symmetric solution of the three-dimensional problem:

$$\begin{cases} -\dfrac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho W(\rho)) + \mu W(\rho) = F(\rho), \quad \mu > 0, \quad \rho > 1, \\ \lim\limits_{\rho\to\infty} \rho^{\frac{3}{2}} W(\rho) = 0, \quad W(1) = g. \end{cases} \tag{5.1}$$

For simplicity, let $g = 0$. We first shift the interval $[1, \infty)$ to $[0, \infty)$ by using the variable transform: $\rho = x + 1$, $W(\rho) = V(x)$, $F(\rho) = G(x)$. Then (5.1) becomes

$$\begin{cases} -\dfrac{1}{(x+1)^2}\partial_x((x+1)^2\partial_x V(x)) + \mu V(x) = G(x), \quad \mu > 0, \quad x > 0, \\ \lim\limits_{x\to\infty} x^{\frac{3}{2}} V(x) = V(0) = 0. \end{cases} \tag{5.2}$$

As mentioned earlier, it is necessary to make the following transformation:

$$V(x) = e^{-\frac{\beta}{2}x}U(x), \quad G(x) = (x+1)^{-2}e^{-\frac{\beta}{2}x}f(x).$$

Then (5.2) is rewritten as

$$\begin{cases} -\partial_x^2 U(x) - \dfrac{1}{x+1}(2-\beta(x+1))\partial_x U(x) + \dfrac{1}{x+1}\Big(\Big(\mu - \dfrac{1}{4}\beta^2\Big)(x+1) + \beta\Big)U(x) \\ \quad = \dfrac{1}{(x+1)^2} f(x), \quad \mu > 0, \quad x > 0, \\ \lim\limits_{x\to\infty} x^{\frac{3}{2}} e^{-\frac{\beta}{2}x} U(x) = U(0) = 0. \end{cases} \tag{5.3}$$

Now, let $\sigma_{\alpha,\beta}(x) = (x+1)^\alpha e^{-\beta x}$, and denote ${}_0H^1_{\sigma_{2,\beta}}(\Lambda) := \{v \in H^1_{\sigma_{2,\beta}}(\Lambda) : u(0) = 0\}$. A weak form of (5.3) is to find $U \in {}_0H^1_{\sigma_{2,\beta}}(\Lambda) \cap L^2_{\sigma_{1,\beta}}(\Lambda)$ such that

$$\widetilde{a}_{\mu,\beta}(U, v) = (f, v)_{\omega_{0,\beta}} \quad \forall v \in {}_0H^1_{\sigma_{2,\beta}}(\Lambda) \cap L^2_{\sigma_{1,\beta}}(\Lambda), \tag{5.4}$$

where the bilinear form

$$\widetilde{a}_{\mu,\beta}(u, v) = (\partial_x u, \partial_x v)_{\sigma_{2,\beta}} + \Big(\mu - \frac{1}{4}\beta^2\Big)(u, v)_{\sigma_{2,\beta}} + \beta(u, v)_{\sigma_{1,\beta}}.$$

One can verify readily that

$$|\widetilde{a}_{\mu,\beta}(u, v)| \le c\big((1+\beta)\|u\|_{1,\sigma_{2,\beta}} + \beta^{\frac{1}{2}}\|u\|_{\sigma_{1,\beta}}\big)\big((1+\beta)\|v\|_{1,\sigma_{2,\beta}} + \beta^{\frac{1}{2}}\|v\|_{\sigma_{1,\beta}}\big), \tag{5.5}$$

and for $\mu > \frac{1}{4}\beta^2$,

$$|\widetilde{a}_{\mu,\beta}(v, v)| \ge c\big(\|v\|^2_{1,\sigma_{2,\beta}} + \beta\|v\|^2_{\sigma_{1,\beta}}\big). \tag{5.6}$$

Hence, if $f \in (H^1_{\sigma_{2,\beta}}(\Lambda) \cap L^2_{\sigma_{1,\beta}}(\Lambda))'$, then (5.4) has a unique solution.

The generalized Laguerre pseudospectral scheme for (5.4) is to seek $u_N \in {}_0\mathbb{P}_N := \{u \in \mathbb{P}_N : u(0) = 0\}$ such that

$$\widetilde{a}_{\mu,\beta,N}(u_N, \phi) = (f, \phi)_{\omega_{0,\beta},R,N} \quad \forall \phi \in {}_0\mathbb{P}_N, \tag{5.7}$$

where

$$\widetilde{a}_{\mu,\beta,N}(u,v) = (\partial_x u, \partial_x v)_{\omega_{2,\beta},R,N} + 2(\partial_x u, \partial_x v)_{\omega_{1,\beta},R,N} + (\partial_x u, \partial_x v)_{\omega_{0,\beta},R,N}$$
$$+\Big(\mu - \frac{1}{4}\beta^2\Big)(u,v)_{\omega_{2,\beta},R,N} + \Big(2\mu - \frac{1}{2}\beta^2 + \beta\Big)(u,v)_{\omega_{1,\beta},R,N}$$
$$+\Big(\mu - \frac{1}{4}\beta^2 + \beta\Big)(u,v)_{\omega_{0,\beta},R,N}.$$

According to (2.22), (5.7) is equivalent to

$$\widetilde{a}_{\mu,\beta}(u_N, \phi) = (\mathcal{I}_{R,N,0,\beta} f, \phi)_{\omega_{0,\beta}} \quad \forall \phi \in {}_0\mathbb{P}_N. \tag{5.8}$$

5.1. A specific orthogonal projection. We next consider a specific orthogonal projection that will be used in numerical analysis of generalized Laguerre pseudospectral method for exterior problems. Let ${}_0H^1_{\omega_{\alpha,\beta}}(\Lambda) := \{v \in H^1_{\omega_{\alpha,\beta}}(\Lambda) : v(0) = 0\}$. Note that $H^1_{\omega_{2,\beta}}(\Lambda) \cap H^1_{\omega_{0,\beta}}(\Lambda) \subseteq H^1_{\sigma_{2,\beta}}(\Lambda) \cap L^2_{\sigma_{1,\beta}}(\Lambda)$. The orthogonal projection ${}_0\Pi^1_{N,\beta} : {}_0H^1_{\omega_{2,\beta}}(\Lambda) \cap H^1_{\omega_{0,\beta}}(\Lambda) \to {}_0\mathbb{P}_N$ is defined by

$$\big(\partial_x({}_0\Pi^1_{N,\beta} v - v), \partial_x \phi\big)_{\sigma_{2,\beta}} + \big({}_0\Pi^1_{N,\beta} v - v, \phi\big)_{\sigma_{2,\beta}} = 0 \quad \forall \phi \in {}_0\mathbb{P}_N. \tag{5.9}$$

In order to analyze approximation error of the previous projection, we need another auxiliary orthogonal projection. To do this, we introduce the space $H^1_{\omega_{2,\beta},\omega_{0,\beta}}(\Lambda)$, equipped with the norm $\|v\|_{1,\omega_{2,\beta},\omega_{0,\beta}} = (\|\partial_x v\|^2_{\omega_{2,\beta}} + \|v\|^2_{\omega_{0,\beta}})^{\frac{1}{2}}$.

The orthogonal projection $\widetilde{P}^1_{N,\beta} : H^1_{\omega_{2,\beta},\omega_{0,\beta}}(\Lambda) \to \mathbb{P}_N$ is defined by

$$\big(\partial_x(\widetilde{P}^1_{N,\beta} v - v), \partial_x \phi\big)_{\omega_{2,\beta}} + \big(\widetilde{P}^1_{N,\beta} v - v, \phi\big)_{\omega_{0,\beta}} = 0 \quad \forall \phi \in \mathbb{P}_N. \tag{5.10}$$

LEMMA 5.1. *If $v \in H^1_{\omega_{2,\beta},\omega_{0,\beta}}(\Lambda) \cap A^r_{1,\beta}(\Lambda)$ and an integer $r \geq 1$, then*

$$\|\widetilde{P}^1_{N,\beta} v - v\|_{1,\omega_{2,\beta},\omega_{0,\beta}} \leq c(\beta N)^{\frac{1-r}{2}} |v|_{A^r_{1,\beta}}.$$

Proof. By projection theorem,

$$\|\widetilde{P}^1_{N,\beta} v - v\|_{1,\omega_{2,\beta},\omega_{0,\beta}} \leq \|\phi - v\|_{1,\omega_{2,\beta},\omega_{0,\beta}} \quad \forall \phi \in \mathbb{P}_N.$$

We take $\phi(x) = P_{N,1,\beta} v(x) - P_{N,1,\beta} v(\frac{1}{\beta}) + v(\frac{1}{\beta})$. Then by (4.11) and (3.1),

$$\|\phi - v\|_{1,\omega_{2,\beta},\omega_{0,\beta}} \leq c\|\partial_x(\phi - v)\|_{\omega_{2,\beta}} = c|P_{N,1,\beta} v - v|_{A^1_{1,\beta}} \leq c(\beta N)^{\frac{1-r}{2}} |v|_{A^r_{1,\beta}}.$$

This completes the proof. ◻

We are ready to estimate $\|{}_0\Pi^1_{N,\beta} v - v\|_{1,\sigma_{2,\beta}}$. We shall use the fact that for $v \in H^1_{\omega_{\alpha,\beta}}(\Lambda)$, $v(0) = 0$ and $\alpha < 1$, we have (see Lemma 2.2 of [12])

$$\|v\|_{\omega_{\alpha,\beta}} \leq c\beta^{-1} \|\partial_x v\|_{\omega_{\alpha,\beta}}. \tag{5.11}$$

LEMMA 5.2. *For any $v \in A^r_{0,\beta}(\Lambda)$ with $v(0) = 0$ and an integer $r \geq 2$,*

$$\|{}_0\Pi^1_{N,\beta} v - v\|_{1,\sigma_{2,\beta}} \leq c(1 + \beta^{-2})(\beta N)^{1-\frac{r}{2}} |v|_{A^r_{0,\beta}}.$$

Proof. By projection theorem,

$$\|{}_0\Pi^1_{N,\beta} v - v\|_{1,\sigma_{2,\beta}} \leq \|\phi - v\|_{1,\sigma_{2,\beta}} \leq c(\|\phi - v\|_{1,\omega_{2,\beta}} + \|\phi - v\|_{1,\omega_{0,\beta}}) \quad \forall \phi \in {}_0\mathbb{P}_N.$$

Taking

$$\phi(x) = \int_0^x \widetilde{P}^1_{N-1,\beta}\partial_\xi v(\xi)\, d\xi \in {}_0\mathbb{P}_N,$$

we have from Lemma 5.1 that

$$\|\partial_x(\phi - v)\|_{\omega_{0,\beta}} = \|\widetilde{P}^1_{N-1,\beta}\partial_x v - \partial_x v\|_{\omega_{0,\beta}} \le c(\beta N)^{1-\frac{r}{2}}|v|_{A^r_{0,\beta}}, \tag{5.12}$$

which, along with (5.11) with $\alpha = 0$, gives

$$\|\phi - v\|_{\omega_{0,\beta}} \le c\beta^{-1}\|\partial_x(\phi - v)\|_{\omega_{0,\beta}} \le c\beta^{-1}(\beta N)^{1-\frac{r}{2}}|v|_{A^r_{0,\beta}}. \tag{5.13}$$

Moreover, thanks to (4.12) with $\alpha = 2$, we have from Lemma 5.1 that

$$\begin{aligned}\|\partial_x(\phi - v)\|_{\omega_{2,\beta}} &= \|\widetilde{P}^1_{N-1,\beta}\partial_x v - \partial_x v\|_{\omega_{2,\beta}} \\ &\le c\beta^{-1}(\|\partial_x(\widetilde{P}^1_{N-1,\beta}\partial_x v - \partial_x v)\|_{\omega_{2,\beta}} + \|\widetilde{P}^1_{N-1,\beta}\partial_x v - \partial_x v\|_{\omega_{0,\beta}}) \\ &\le c\beta^{-1}(\beta N)^{1-\frac{r}{2}}|v|_{A^r_{0,\beta}}.\end{aligned} \tag{5.14}$$

Furthermore, using (4.12), (5.13), and (4.13) leads to

$$\|\phi - v\|_{\omega_{2,\beta}} \le c\beta^{-1}(\|\partial_x(\phi - v)\|_{\omega_{2,\beta}} + \|\phi - v\|_{\omega_{0,\beta}}) \le c\beta^{-2}(\beta N)^{1-\frac{r}{2}}|v|_{A^r_{0,\beta}}. \tag{5.15}$$

Finally, a combination of (5.12)–(5.15) leads to the desired result. $\square$

5.2. Convergence analysis. Let $U_N = {}_0\Pi^1_{N,\beta}U$. Then by (5.4) and (5.9),

$$\widetilde{a}_{\mu,\beta}(U_N, \phi) = -\widetilde{G}(\phi) + (\mathcal{I}_{R,N,0,\beta}f, \phi)_{\omega_{0,\beta}} \quad \forall \phi \in {}_0\mathbb{P}_N, \tag{5.16}$$

where

$$\widetilde{G}(\phi) = \Big(\mu - \frac{1}{4}\beta^2 - 1\Big)(U - U_N, \phi)_{\sigma_{2,\beta}} + \beta(U - U_N, \phi)_{\sigma_{1,\beta}} + (\mathcal{I}_{R,N,0,\beta}f - f, \phi)_{\omega_{0,\beta}}.$$

Set $\widetilde{U}_N = u_N - U_N$. Then subtracting (5.16) from (5.8) yields

$$\widetilde{a}_{\mu,\beta}(\widetilde{U}_N, \phi) = \widetilde{G}(\phi) \quad \forall \phi \in {}_0\mathbb{P}_N. \tag{5.17}$$

Taking $\phi = \widetilde{U}_N$ in the previous formula and using (5.6), we obtain

$$\|\widetilde{U}_N\|^2_{1,\sigma_{2,\beta}} + \beta\|\widetilde{U}_N\|^2_{\sigma_{1,\beta}} \le c|\widetilde{G}(\widetilde{U}_N)|. \tag{5.18}$$

Thus, it suffices to estimate $|\widetilde{G}(\widetilde{U}_N)|$. For simplicity, we will use the following notation:

$$\begin{aligned}\widetilde{B}^{(1)}_{N,\beta,r}(v) &= c(\beta^2+1)^2(1+\beta^{-2})^2(\beta N)^{2-r}|v|^2_{A^r_{0,\beta}}, \\ \widetilde{B}^{(2)}_{N,\beta,r}(v) &= c(1+\beta^{-2})^2\beta(\beta N)^{2-r}|v|^2_{A^r_{0,\beta}}, \\ \widetilde{B}^{(3)}_{N,\beta,s}(v) &= c\beta^{-1}(\beta N)^{1-s}\Big(\beta^{-2}\|\partial_x^s v\|^2_{\omega_{s-1,\beta}} + (1+\beta^{-1})\ln N|v|^2_{A^s_{0,\beta}}\Big).\end{aligned}$$

By virtue of Theorem 3.7 and Lemma 5.2, for integers $r \ge 2$ and $s \ge 1$,

$$|\widetilde{G}(\widetilde{U}_N)| \le \widetilde{B}^{(1)}_{N,\beta,r}(U) + \widetilde{B}^{(2)}_{N,\beta,r}(U) + \widetilde{B}^{(3)}_{N,\beta,s}(f) + \frac{1}{2}\|\widetilde{U}_N\|^2_{\sigma_{2,\beta}} + \frac{\beta}{2}\|\widetilde{U}_N\|^2_{\sigma_{1,\beta}}.$$

Plugging the previous formula into (5.18) leads to an estimate for $\|\widetilde{U}_N\|^2_{1,\sigma_{2,\beta}} + \beta\|\widetilde{U}_N\|^2_{\sigma_{1,\beta}}$. Finally, we use Lemma 5.2 again to reach the following conclusion.

THEOREM 5.3. *Let U and u_N be the solutions of* (5.4) *and* (5.7)*, respectively, and $\mu > \frac{1}{4}\beta^2$. If $U \in A^r_{0,\beta}(\Lambda)$ with $U(0) = 0$, and $f \in A^s_{0,\beta}(\Lambda)$ and $\partial^s_x f \in L^2_{\omega_{s-1,\beta}}(\Lambda)$ with integers $r \geq 2$ and $s \geq 1$, then*

$$\|U - u_N\|^2_{1,\sigma_{2,\beta}} + \beta\|U - u_N\|^2_{\sigma_{1,\beta}} \leq \widetilde{B}^{(1)}_{N,\beta,r}(U) + \widetilde{B}^{(2)}_{N,\beta,r}(U) + \widetilde{B}^{(3)}_{N,\beta,s}(f).$$

6. Numerical results. We present numerical results to illustrate the efficiency of the proposed schemes.

6.1. The scheme (4.8). We first take a look at the matrix form of the system (4.8). We take the base functions $\psi_j(\rho) = \mathcal{L}^{(1,\beta)}_j(\rho)$ and let $\mathbb{P}_N = \text{span}\{\psi_0, \psi_1, \ldots, \psi_N\}$. By (2.5) and (2.3), we have

$$\psi_j(\rho) = \frac{1}{\beta}\Big(\partial_\rho \mathcal{L}^{(1,\beta)}_j(\rho) - \partial_\rho \mathcal{L}^{(1,\beta)}_{j+1}(\rho)\Big) = -\mathcal{L}^{(2,\beta)}_{j-1}(\rho) + \mathcal{L}^{(2,\beta)}_j(\rho).$$

This fact together with (2.3) and (2.6) leads to

$$a_{jk} := (\partial_\rho\psi_k, \partial_\rho\psi_j)_{\omega_{2,\beta}} = \beta^2\gamma^{(2,\beta)}_{k-1}\delta_{j,k}, \quad m_{jk} := (\psi_k, \psi_j)_{\omega_{1,\beta}} = \gamma^{(1,\beta)}_k\delta_{j,k},$$
$$s_{jk} := (\psi_k, \psi_j)_{\omega_{2,\beta}} = \begin{cases} -\gamma^{(2,\beta)}_{k-1}, & j = k-1, \\ \gamma^{(2,\beta)}_{k-1} + \gamma^{(2,\beta)}_k, & j = k, \\ -\gamma^{(2,\beta)}_k, & j = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we set

$$\begin{aligned} &u_N(\rho) = \sum_{j=0}^N \hat{u}_j\psi_j(\rho), \quad \mathbf{u} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_N)^T, \quad f_j = (f, \psi_j)_{\omega_{2,\beta},G,N}, \\ &\mathbf{f} = (f_0, f_1, \ldots, f_N)^T, \quad A = (a_{jk})_{0\leq j,k\leq N}, \quad M = (m_{jk})_{0\leq j,k\leq N}, \\ &S = (s_{jk})_{0\leq j,k\leq N}. \end{aligned} \tag{6.1}$$

Then the system (4.9) becomes

$$\Big(A + \Big(\mu - \frac{\beta^2}{4}\Big)S + \beta M\Big)\mathbf{u} = \mathbf{f}. \tag{6.2}$$

It is seen that this system is symmetric, tridiagonal, and easy to be inverted.

We now present some numerical results using the previous scheme to solve (4.1) with spherically symmetric solution $W(\rho)$. Basically, we find $u_N(\rho)$ from the system (6.2), and then evaluate the numerical solution by $w_N(\rho) = e^{-\frac{\beta}{2}\rho}u_N(\rho)$. In the following computations, let $\mu = 5$ in (4.8).

Example 1. We take the test function $W(\rho) = e^{-\gamma\rho}\sin h\rho$, with $\gamma > 0$, which decays exponentially at infinity. The corresponding solution of formula (4.4) is $U(\rho) = e^{(\beta/2-\gamma)\rho}\sin h\rho$. We measure the errors in two ways:

(i) maximum pointwise error:

$$\max_{0\leq j\leq N}\Big|\Big(W(\xi^{1,\beta}_{G,N,j}) - w_N(\xi^{1,\beta}_{G,N,j})\Big)\xi^{1,\beta}_{G,N,j}\Big| \sim \sup_{\rho\in\Lambda}|\rho(W(\rho) - w_N(\rho))|;$$

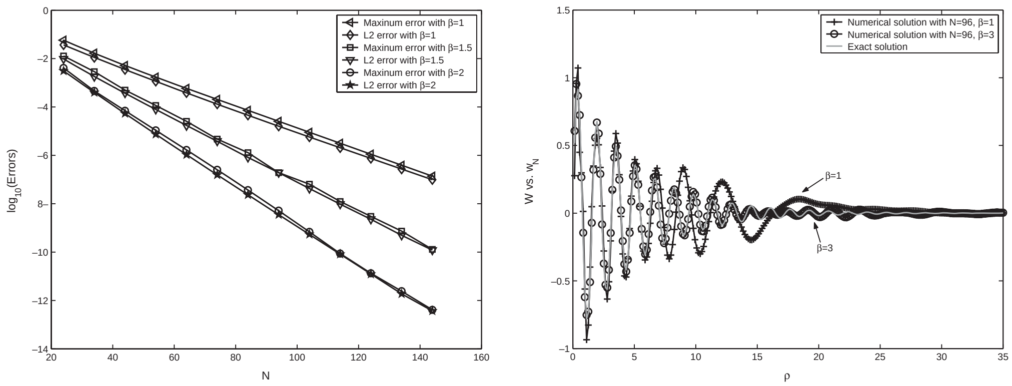

FIG. 6.1. *Convergence rate: Example* 1 *with* $\gamma = 1$ *and* $h = 3$ *on the left. Generalized Laguerre approximation: Example* 1 *with* $\gamma = 0.2$ *and* $h = 4$ *on the right.*

(ii) discrete L^2-error:

$$\|W - w_N\|_N := \|U - u_N\|_{\omega_{1,\beta},G,N} \sim \|W - w_N\|_{\hat{\omega}_1}.$$

In the left part of Figure 6.1, we plot the $\log_{10}$ of maximum error and the $\log_{10}$ of L^2-error against various N for $\gamma = 1,\ h = 3$, and different β. As predicted in Theorem 4.2 and Remark 4.1, the approximate solution will converge faster than any algebraic power, which is confirmed by the error behaviors (like $e^{-cN}, c > 0$) as shown in the figure. We also see that for fixed N, the scheme with $\beta = 2$ or $\beta = 1.5$ produces better numerical results than that with $\beta = 1$ (the usual generalized Laguerre approximation).

To see more clearly the role of β, we compare in the right part of Figure 6.1 the exact solution with $\gamma = 0.2$ and $h = 4$ with the numerical solution obtained by our pseudospectral scheme with $N = 96$ and $\beta = 1, 3$. Notice that the approximation solution with $\beta = 1$ exhibits an observable error, while the numerical solution with $\beta = 3$ is virtually indistinguishable with the exact solution. This example demonstrates that a suitable choice of the parameter β can raise the accuracy, and also enhance greatly the resolution capabilities of the generalized Laguerre approximations.

Example 2. We take $W(\rho) = \frac{\rho}{(\rho+1)^k}$ with $k > 1$, which decays algebraically at infinity. It is clear that $\rho^{\frac{3}{2}} W(\rho) \to 0$, as $\rho \to \infty$, if $k > \frac{5}{2}$.

In Figure 6.2, we plot the $\log_{10}$ of L^2-errors vs. $\sqrt{N}$ for different k and β. We see that the convergence rates are of order $O(e^{-c\sqrt{N}})$ for all cases, which are somewhat better than those predicted in Theorem 4.2 and Remark 4.1 (no more than order k). We also observe that for larger N, better numerical result can be obtained by choosing suitable $\beta < 1$ if the solution decays slowly (cf. the left part of Figure 6.2, where $W(\rho) = O(\rho^{-1.51})$), while conversely for the solution decaying very fast (cf. the right part of Figure 6.2, where $W(\rho) = O(\rho^{-4})$).

Example 3. We take $W(\rho) = \frac{\sin h\rho}{(1+\rho)^k}$ with $k > 0$, which decays algebraically with oscillation.

In Figure 6.3, we plot the $\log_{10}$ of L^2-errors vs. $\log_{10} N$. In the left part, we take $h = 3,\ k = 4$, and $\beta = 1, 2, 3, 4$, while in the right part, we fix $\beta = 2$ and $h = 3$, and test different $k = 3, 4, 5$. It is clear that in all cases, the errors decay at certain

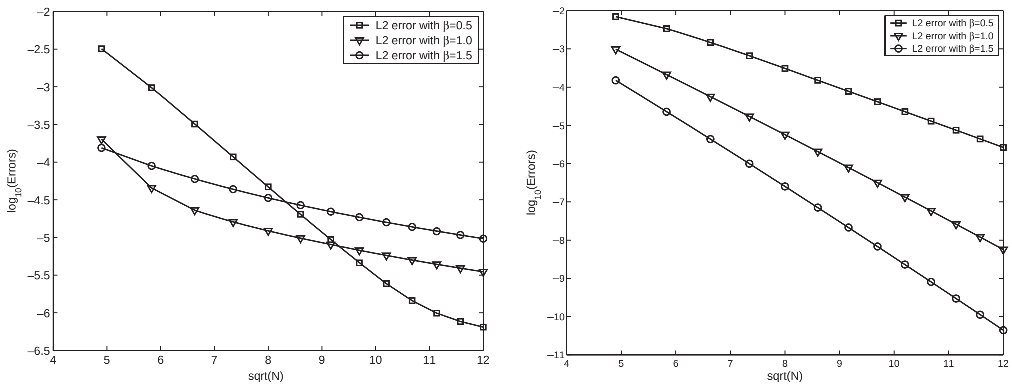

FIG. 6.2. *Convergence rate of generalized Laguerre pseudospectral method: Example* 2 *with* $k = 2.51$ *on the left; Example* 2 *with* $k = 5$ *on the right.*

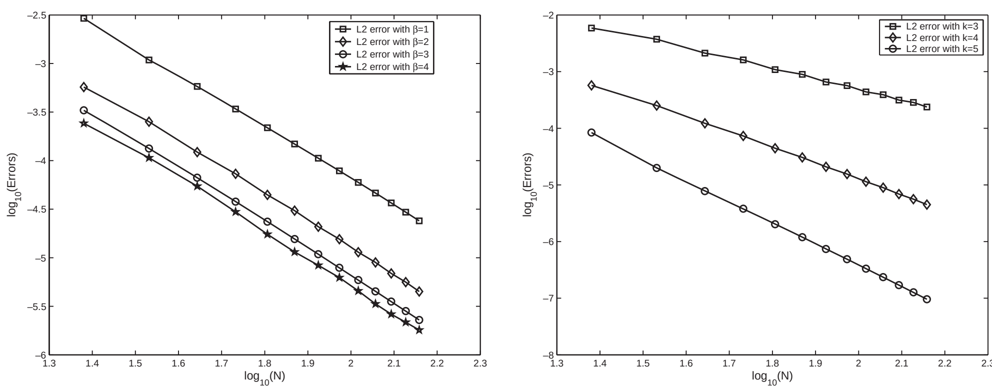

FIG. 6.3. *Convergence rate of generalized Laguerre pseudospectral method: Example* 3 *with* $h = 3$ *and* $k = 4$ *on the left; Example* 3 *with* $h = 3$ *and* $\beta = 2$ *on the right.*

algebraic rate. Once again, we see from the left part of this figure that a suitable parameter β can produce better numerical results. On the other hand, the right part shows that the faster the exact solution decays, the smaller the numerical errors would be. The previous facts coincide again well with our theoretical results.

6.2. The scheme (5.8). We next describe an efficient implementation for scheme (5.8). Set $\psi_j(x) := \mathcal{L}_j^{(0,\beta)}(x) - \mathcal{L}_{j+1}^{(0,\beta)}(x)$, $j \geq 0$, $\beta > 0$. Clearly, $\psi_j(0) = 0$. Hence, ${}_0\mathbb{P}_N = \mathrm{span}\{\psi_0, \psi_1, \ldots, \psi_{N-1}\}$. We now study the structures of the corresponding matrices. Thanks to (2.5), we have $\partial_x \psi_j(x) = \beta \mathcal{L}_j^{(0,\beta)}(x)$, which, along with (2.3), implies that $(1+x)^2 \partial_x \psi_j(x)$ is a linear combination of $\mathcal{L}_l^{(0,\beta)}$, $j-2 \leq l \leq j+2$. This fact with (2.3), (2.5), and (2.6) leads to

$$\begin{aligned} &a_{jk} := (\partial_x \psi_k, \partial_x \psi_j)_{\sigma_{2,\beta}} = 0 \quad \text{if} \quad |j-k| > 2, \\ &b_{jk} := (\psi_k, \psi_j)_{\sigma_{2,\beta}} = 0 \quad \text{if} \quad |j-k| > 3, \\ &c_{jk} := (\psi_k, \psi_j)_{\sigma_{1,\beta}} = 0 \quad \text{if} \quad |j-k| > 2. \end{aligned} \tag{6.3}$$

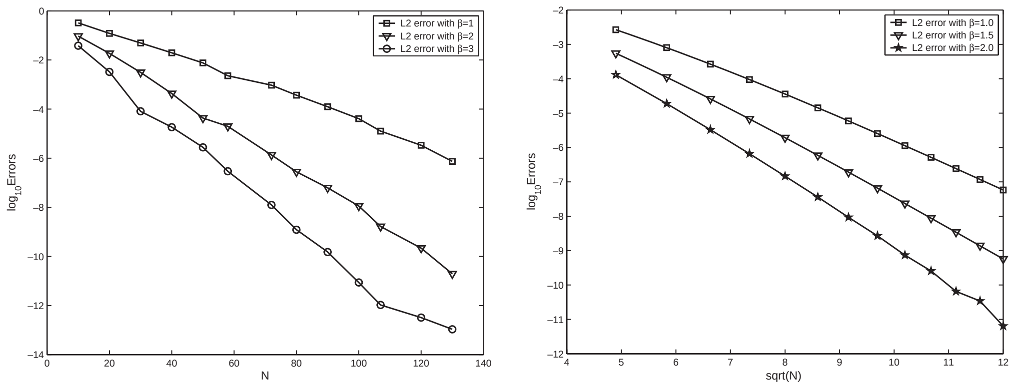

FIG. 6.4. *Convergence rate: Example* 4 *with* $\gamma = 1$ *and* $h = 3$ *on the left; Example* 5 *with* $h = 1$ *and* $k = 5$ *on the right.*

By setting

$$
\begin{aligned}
&u_N = \sum_{j=0}^{N-1} \hat{u}_j \psi_j(\rho), \quad \mathbf{u} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-1})^{\mathrm{T}},\\
&f_j = (f, \psi_j)_{\omega_{0,\beta},R,N}, \quad \mathbf{f} = (f_0, f_1, \dots, f_{N-1})^{\mathrm{T}},\\
&A = (a_{jk})_{0\le j,k\le N-1}, \quad B = (b_{jk})_{0\le j,k\le N-1}, \quad C = (c_{jk})_{0\le j,k\le N-1},
\end{aligned}
\tag{6.4}
$$

the system (5.8) becomes

$$
\Big(A + \Big(\mu - \frac{\beta^2}{4}\Big)B + \beta C\Big)\mathbf{u} = \mathbf{f}. \tag{6.5}
$$

The coefficient matrix is symmetric and has only several nonvanishing diagonals. Moreover, the nonzero entries can be determined explicitly by using properties of generalized Laguerre polynomials as shown in section 2.

We present below two numerical examples to show the efficiency of generalized Laguerre pseudospectral methods for exterior problems. Let $\mu = 5$ in (5.7).

Example 4. We take the test function $W(\rho) = e^{-\gamma(\rho-1)} \sin h(\rho - 1)$ with $\gamma > 0$ and $\rho \ge 1$, which decays exponentially at infinity. The corresponding solution of (5.3) is $U(\rho) = e^{(\beta/2-\gamma)\rho} \sin h\rho$. We denote the discrete L^2-error by $\|W - w_N\|_N := \|U - u_N\|_{\omega_{0,\beta},R,N}$.

In the left part of Figure 6.4, we plot the $\log_{10}$ of L^2-error against various N for $\gamma = 1,\ h = 3$, and different β. We observe a convergence rate of order $O(e^{-cN})$, as predicted in Theorem 5.3. Moreover, for fixed N, the scheme with $\beta = 3$ or $\beta = 2$ produces better numerical results than that with $\beta = 1$.

Example 5. We take $W(\rho) = \frac{\sin(h(\rho-1))}{\rho^k}$ with $k > 0$ and $\rho \ge 1$, which decays algebraically with oscillation. It is clear that $\rho^{\frac{3}{2}} W(\rho) \to 0$, as $\rho \to \infty$, if $k > \frac{3}{2}$.

In the right part of Figure 6.4, we plot the $\log_{10}$ of L^2-error vs. $\sqrt{N}$ for $h = 1,\ k = 5$, and different β. It is seen that the convergence rates are of order $O(e^{-c\sqrt{N}})$, which are somewhat better than what were predicted in Theorem 5.3 (no more than order k). Once again, the error behaviors confirm that a suitable choice of β gives better numerical results than that obtained from the usual generalized Laguerre approximation ($\beta = 1$).

7. Concluding remarks. In this paper, we established a set of results on generalized Laguerre–Gauss-type interpolation in nonuniformly weighted Sobolev spaces with the weight function $\omega_{\alpha,\beta}(x) = x^\alpha e^{-\beta x},\ \alpha > -1, \beta > 0$, which provided us useful tools in developing and analyzing generalized pseudospectral methods for a variety of problems in unbounded domains.

Several advantages justified our choice of working on the orthogonal system $\{\mathcal{L}_l^{(\alpha,\beta)}(x)\}$ with general parameters $\alpha > -1, \beta > 0$.

- The parameter α played an essential part in forming the pseudospectral schemes, which was chosen to agree with the degree of singular coefficients of leading terms in underlying equations. For instance, we take $\alpha = 2$ for three-dimensional problems as in (4.1) and (5.1), while we should take $\alpha = 1$ for two-dimensional problems.
- The adjustable parameter β offers great flexibility to match various asymptotic behaviors of the solutions at infinity. In fact, a suitable choice of β depends on certain coefficients which determine the asymptotic behaviors of solutions such as the parameter μ in (4.1) and (5.1).
- The parameter β somehow played a role similar to a scaling factor, which could improve the numerical resolution. But they are not exactly the same. Indeed, in the scaling method with variable transformation, one approximates the function $v(\beta x)$ by the basis $\{\mathcal{L}_l^{(\alpha)}(\beta x)\}$. However, we approximate the function $v(x)$ directly.
- As shown in sections 4 and 5, we could always choose a suitable value of β to guarantee the well-posedness of our Galerkin formulation, provided that some conditions are fulfilled.
- From theoretical point of view, our analysis included usual Laguerre ($\alpha = 0$ and $\beta = 1$) and standard generalized Laguerre ($\alpha > -1$ and $\beta = 1$) approximations as special cases. Moreover, our estimates improved the previously published results for the special case $\alpha = 0$ and $\beta = 1$ (cf. [19]). Roughly speaking, the factor $N^\gamma, \gamma > 0$ appearing in the upper bound of the interpolation error of [19] is now replaced by $\ln N$.

In this paper, our pseudospectral method was designed for transformed equations (cf. (4.4) and (5.3)). We may also take the generalized Laguerre functions $\widehat{\mathcal{L}}_l^{(\alpha,\beta)}(x) = e^{-\frac{\beta}{2}x}\mathcal{L}_l^{(\alpha,\beta)}(x)$ as the base functions that are mutually orthogonal with the weight $\hat{\omega}_\alpha(x) = x^\alpha$. In this case, the weights of Gauss quadrature, $\hat{\omega}_{Z,N,j}^{(\alpha,\beta)} = e^{\beta\xi_{Z,N,j}^{(\alpha,\beta)}}\omega_{Z,N,j}^{(\alpha,\beta)}$, $Z = G, R,\ 0 \le j \le N$. Then, the pseudospectral scheme for (4.4) is to find $w_N \in \widehat{\mathbb{P}}_N := \{\phi : \phi = e^{-\frac{\beta}{2}x}\psi \quad \forall \psi \in \mathbb{P}_N\}$ such that

$$(\partial_x w_N, \partial_x \phi)_{\hat{\omega}_2,G,N} + \mu(w_N, \phi)_{\hat{\omega}_2,G,N} = (f, \phi)_{\hat{\omega}_2,G,N} \quad \forall \phi \in \widehat{\mathbb{P}}_N,$$

where $(\cdot,\cdot)_{\hat{\omega}_2,G,N}$ is the corresponding discrete inner product.

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