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Mixed Fourier–Jacobi spectral method

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Abstract

This paper is for mixed Fourier–Jacobi approximation and its applications to numerical solutions of semi-periodic singular problems, semi-periodic problems on unbounded domains and axisymmetric domains, and exterior problems. The stability and convergence of proposed spectral schemes are proved. Numerical results demonstrate the efficiency of this new approach.

Keywords: Mixed Fourier–Jacobi spectral method; Applications

1. Introduction

In this paper, we investigate the mixed Fourier–Jacobi spectral method and its applications. This work is motivated by several facts. Firstly we need to solve semi-periodic problems numerically when we study the boundary layer, the flow past a suddenly heated vertical plate and other related problems, see [9,11,12]. We usually use the Fourier spectral-finite difference method, the Fourier spectral-finite element method, and the mixed Fourier–Legendre and Fourier–Chebyshev spectral methods, see [2,5,8–10,12]. However in many cases, we have to consider semi-periodic singular problems. In this case, it is natural to use the Jacobi approximation developed in [3,4,6],

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in the non-periodic directions. The second motivation is numerical simulation of semi-periodic problems on unbounded domains. We may impose certain artificial boundary conditions in the non-periodic directions. However, it causes additional errors. In particular, it is difficult to apply this trick to nonlinear problems. Some authors developed the mixed Fourier–Laguerre and Fourier–Hermite spectral methods. But both of them require quadratures over unbounded domains. In fact, we could reform such problems and then use the mixed Fourier–Jacobi spectral method for the resulting ones. Finally, the mixed Fourier–Jacobi spectral method is also motivated by some problems on axisymmetric domains, see [1], and exterior problems.

This paper is organized as follows. In the next section, we recall some basic results on one-dimensional Jacobi approximation. In Section 3, we establish some results on the mixed Fourier–Jacobi approximation which play important roles in the analysis of mixed Fourier–Jacobi spectral method. Section 4 is for some applications. We first consider a linear model problem, and then deal with the nonlinear Klein–Gordon equation. The stability and convergence of proposed schemes are proved. Some other applications are discussed. We also present some numerical results showing the high accuracy of this new approach.

2. Preliminaries

Let $\Lambda = (-1, 1)$ and $\chi(x)$ be a certain weight function in the usual sense. For $1 \leqslant p \leqslant \infty$, $L^p_\chi(\Lambda) = \{v \mid v \text{ is measurable and } \|v\|_{L^p_\chi} < \infty\}$, with the norm

$$\|v\|_{L^p_\chi} = \begin{cases} \left(\int_\Lambda |v(x)|^p \chi(x)\,dx\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, we denote by $(u, v)_\chi$ and $\|v\|_\chi$ the inner product and the norm of space $L^2_\chi(\Lambda)$. Furthermore, for non-negative integer m, $H^m_\chi(\Lambda) = \{v \mid \|v\|_{m,\chi} < \infty\}$, equipped with the seminorm $|v|_{m,\chi} = \|\partial_x^m v\|_\chi$, and the norm $\|v\|_{m,\chi} = (\sum_{k=0}^m |v|^2_{k,\chi})^{1/2}$. For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ by the space interpolation. $H^r_{0,\chi}(\Lambda)$ is the closure in $H^r_\chi(\Lambda)$ of the set of all infinitely differentiable functions with compact support in Λ. We omit r for $r = 0$ and omit χ for $\chi \equiv 1$, in the notations.

We denote by $J_l^{(\alpha,\beta)}(x)$ the Jacobi polynomials of degree l. Let $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$. The set of Jacobi polynomials is an $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

For any positive integer M, $\mathcal{P}_M(\Lambda)$ stands for the set of all algebraic polynomials of degree at most M. Further, ${}_0\mathcal{P}_M(\Lambda) = \{v \mid v \in \mathcal{P}_M(\Lambda),\ v(-1) = 0\}$ and $\mathcal{P}^0_M(\Lambda) = \{v \mid v \in \mathcal{P}_M(\Lambda),\ v(-1) = v(1) = 0\}$. We shall use two inverse inequalities in $\mathcal{P}_M(\Lambda)$, stated below. They are proved in [4]. In the sequel, c denotes a generic positive constant independent of any function and M.

Lemma 2.1. *For any $\phi \in \mathcal{P}_M(\Lambda)$ and $1 \leqslant p \leqslant q \leqslant \infty$,*

$$\|\phi\|_{L^q_{\chi^{(\alpha,\beta)}},\Lambda} \leqslant cM^{\sigma(\alpha,\beta)(1/p-1/q)}\|\phi\|_{L^p_{\chi^{(\alpha,\beta)}},\Lambda},$$

where $\sigma(\alpha,\beta) = 2\max(\alpha,\beta) + 2$ if $\max(\alpha,\beta) \geqslant -\frac{1}{2}$, and $\sigma(\alpha,\beta) = 1$ otherwise.

Lemma 2.2. *For any $\phi \in \mathcal{P}_M(\Lambda)$ and $r \geqslant 0$,*

$$\|\phi\|_{r,\chi^{(\alpha,\beta)},\Lambda} \leqslant cM^{2r}\|\phi\|_{\chi^{(\alpha,\beta)},\Lambda}.$$

If, in addition, $\alpha, \beta > r-1$, *then*

$$\|\phi\|_{r,\chi^{(\alpha,\beta)},\Lambda} \leqslant cM^r \|\phi\|_{\chi^{(\alpha-r,\beta-r)},\Lambda}.$$

We shall use also several orthogonal projections in the next section. For technical reasons, we define the non-uniformly weighted Sobolev space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$. For any integer $r \geqslant 0$,

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \big\{v \mid v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},A,\Lambda} < \infty\big\},$$

equipped with the following semi-norm and norm:

$$|v|_{r,\chi^{(\alpha,\beta)},A,\Lambda} = \big\|\partial_x^r v\big\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}, \qquad \|v\|_{r,\chi^{(\alpha,\beta)},A,\Lambda} = \left(\sum_{k=0}^{r} |v|^2_{k,\chi^{(\alpha,\beta)},A,\Lambda}\right)^{1/2}.$$

For any real number $r > 0$, we define the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ by space interpolation. Further, let $H^r_{\chi^{(\alpha,\beta)},*}(\Lambda) = \{v \mid \partial_x v \in H^{r-1}_{\chi^{(\alpha,\beta)},A}(\Lambda)\}$ with the semi-norm $|v|_{r,\chi^{(\alpha,\beta)},*,\Lambda} = |\partial_x v|_{r-1,\chi^{(\alpha,\beta)},A,\Lambda}$ and the norm $\|v\|_{r,\chi^{(\alpha,\beta)},*,\Lambda} = \|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},A,\Lambda}$.

Let $(u,v)_{\chi^{(\alpha,\beta)},\Lambda}$ be the inner product of the space $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$. The orthogonal projection $P_{M,\alpha,\beta,\Lambda} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_M(\Lambda)$ is defined by

$$(P_{M,\alpha,\beta,\Lambda} v - v, \phi)_{\chi^{(\alpha,\beta)},\Lambda} = 0, \quad \forall \phi \in \mathcal{P}_M(\Lambda).$$

We have the following basic result, see [4, p. 387].

Lemma 2.3. *For any* $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ *and integer* $r \geqslant 0$,

$$\|P_{M,\alpha,\beta,\Lambda} v - v\|_{\chi^{(\alpha,\beta)},\Lambda} \leqslant cM^{-r} |v|_{r,\chi^{(\alpha,\beta)},A,\Lambda}.$$

Next, we take $\alpha, \beta, \gamma, \delta > -1$ and introduce the spaces $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \leqslant \mu \leqslant 1$. For $\mu = 0$, $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \big\{v \mid v \text{ is measurable and } \|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} < \infty\big\}$$

equipped with the following semi-norm and norm:

$$|v|_{1,\alpha,\beta,\gamma,\delta,\Lambda} = \|\partial_x v\|_{\chi^{(\alpha,\beta)},\Lambda}, \qquad \|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} = \big(|v|^2_{1,\alpha,\beta,\gamma,\delta,\Lambda} + \|v\|^2_{\chi^{(\gamma,\delta)},\Lambda}\big)^{1/2}.$$

For $0 < \mu < 1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation. In particular,

$${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \big\{v \,\big|\, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0\big\}.$$

Let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)},\Lambda} + (u,v)_{\chi^{(\gamma,\delta)},\Lambda}.$$

The orthogonal projection $P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_M(\Lambda)$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}\big(P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda} v - v, \phi\big) = 0, \quad \forall \phi \in \mathcal{P}_M(\Lambda).$$

The orthogonal projection ${}_0P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} : {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathcal{P}_M(\Lambda)$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}\big({}_0P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} v - v, \phi\big) = 0, \quad \forall \phi \in {}_0\mathcal{P}_M(\Lambda).$$

The orthogonal projection $\tilde{P}^{1,0}_{M,\alpha,\beta,\Lambda}: H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^0_M(\Lambda)$ is defined by

$$\left(\partial_x\left(\tilde{P}^{1,0}_{M,\alpha,\beta,\Lambda} v - v\right), \partial_x \phi\right)_{\chi^{(\alpha,\beta)},\Lambda} = 0, \quad \forall \phi \in \mathcal{P}^0_M(\Lambda).$$

We shall use the following results, see [7].

Lemma 2.4. *If $\alpha \leqslant \gamma + 2$ and $\beta \leqslant \delta + 2$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and integer $r \geqslant 1$,*

$$\left\| P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} v - v \right\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} \leqslant cM^{1-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

If, in addition, $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 1$, then for $0 \leqslant \mu \leqslant 1$,

$$\left\| P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} v - v \right\|_{\mu,\alpha,\beta,\gamma,\delta,\Lambda} \leqslant cM^{\mu-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

Lemma 2.5. *If $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 2$, $0 < \alpha < 1$, $\beta < 1$ or $\alpha \leqslant \gamma + 2$, $\beta \leqslant 0$, $\delta \geqslant 0$, then for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and integer $r \geqslant 1$,*

$$\left\| {}_0P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} v - v \right\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} \leqslant cM^{1-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

If, in addition, $\beta \leqslant \delta + 1$, then for $0 \leqslant \mu \leqslant 1$,

$$\left\| {}_0P^1_{M,\alpha,\beta,\gamma,\delta,\Lambda} v - v \right\|_{\mu,\alpha,\beta,\gamma,\delta,\Lambda} \leqslant cM^{\mu-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

Lemma 2.6. *If $-1 < \alpha, \beta < 1$, then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,*

$$\left\| \tilde{P}^{1,0}_{M,\alpha,\beta,\Lambda} v - v \right\|_{1,\chi^{(\alpha,\beta)},\Lambda} \leqslant cM^{1-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

If, in addition, $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, then for all $0 \leqslant \mu \leqslant 1$,

$$\left\| \tilde{P}^{1,0}_{M,\alpha,\beta,\Lambda} v - v \right\|_{\mu,\chi^{(\alpha,\beta)},\Lambda} \leqslant cM^{\mu-r} |v|_{r,\chi^{(\alpha,\beta)},*,\Lambda}.$$

3. Mixed Fourier–Jacobi approximation

In this section, we study the mixed Fourier–Jacobi approximation. Let $x = (x_1, x_2)$, $\Lambda_1 = \{x_1 \mid -1 < x_1 < 1\}$, $\Lambda_2 = \{x_2 \mid 0 < x_2 < 2\pi\}$ and $\Omega = \Lambda_1 \times \Lambda_2$. For $r \geqslant 0$, we define the weighted spaces $H^r_\chi(\Omega)$ and $H^r_{0,\chi}(\Omega)$ in the usual way. For simplicity, we denote their semi-norm and norm by $|v|_{r,\chi}$ and $\|v\|_{r,\chi}$, respectively. $\|v\|_{L^p_\chi}$ denotes the norm of $L^p_\chi(\Omega)$, $1 \leqslant p \leqslant \infty$. In addition, $(u, v)_{r,\chi}$ denotes the inner product of $H^r_\chi(\Omega)$ for integer r. We omit r or χ in the notations when $r = 0$ or $\chi \equiv 1$, respectively.

Let M and N be any positive integers. $\mathcal{V}_N(\Lambda_2)$ is the set of all real-valued trigonometric polynomials of degree at most N on Λ_2. $V_{M,N} = \mathcal{P}_M(\Lambda_1) \otimes \mathcal{V}_N(\Lambda_2)$, ${}_0V_{M,N} = {}_0\mathcal{P}_M(\Lambda_1) \otimes \mathcal{V}_N(\Lambda_2)$ and $V^0_{M,N} = \mathcal{P}^0_M(\Lambda_1) \otimes \mathcal{V}_N(\Lambda_2)$. We first establish two inverse inequalities in $V_{M,N}$.

Theorem 3.1. *For any $\phi \in V_{M,N}$ and $2 \leqslant p \leqslant \infty$,*

$$\|\phi\|_{L^p_{\chi^{(\gamma,\delta)}}} \leqslant c\left(M^{\sigma(\gamma,\delta)} N\right)^{1/2-1/p} \|\phi\|_{\chi^{(\gamma,\delta)}}.$$

Proof. Let

$$\phi_l(x_1) = \frac{1}{2\pi} \int_{\Lambda_2} \phi(x_1, x_2) e^{-ilx_2} \, dx_2. \tag{3.1}$$

Clearly $\phi_l \in \mathcal{P}_M(\Lambda_1)$. By the orthogonality of trigonometric functions,

$$\|\phi\|^2_{\chi^{(\gamma,\delta)}} = 2\pi \sum_{|l|\leqslant N} \|\phi_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}.$$

The above with Lemma 2.1 leads to

$$\|\phi\|_{L^\infty(\Omega)} \leqslant \sum_{|l|\leqslant N} \|\phi_l\|_{\infty,\Lambda_1} \leqslant cM^{\frac{1}{2}\sigma(\gamma,\delta)} \sum_{|l|\leqslant N} \|\phi_l\|_{\chi^{(\gamma,\delta)},\Lambda_1} \leqslant c\big(M^{\sigma(\gamma,\delta)}N\big)^{1/2}\|\phi\|_{\chi^{(\gamma,\delta)}}.$$

Accordingly

$$\|\phi\|^p_{L^p_{\chi^{(\gamma,\delta)}}} \leqslant \|\phi\|^{p-2}_{L^\infty}\|\phi\|^2_{\chi^{(\gamma,\delta)}} \leqslant c\big(M^{\sigma(\gamma,\delta)}N\big)^{p/2-1}\|\phi\|^p_{\chi^{(\gamma,\delta)}}. \qquad \square$$

We now establish another inverse inequality. Let $\alpha_i, \beta_i > -1$, $i = 1, 2$, and $\chi^{(\alpha_1,\beta_1)}(x) = \chi^{(\alpha_1,\beta_1)}(x_1)$, $\chi^{(\alpha_2,\beta_2)}(x) = \chi^{(\alpha_2,\beta_2)}(x_1)$. Further, let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, and

$$H^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega) = L^2\big(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)\big) \cap H^s\big(\Lambda_2; L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1)\big)$$

with the norm

$$\|v\|_{H^{r,s}_{\chi^{(\alpha,\beta)}}} = \Big(\|v\|^2_{L^2(\Lambda_2;H^r_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1))} + \|v\|^2_{H^s(\Lambda_2;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1))}\Big)^{1/2}.$$

Theorem 3.2. *For any $\phi \in V_{M,N}$ and $r, s \geqslant 0$,*

$$\|\phi\|_{H^{r,s}_{\chi^{(\alpha,\beta)}}} \leqslant c\big(M^{2r} + N^s\big)\|\phi\|_{\chi^{(\eta_1,\eta_2)}},$$

where $\eta_1 = \min(\alpha_1, \alpha_2)$ and $\eta_2 = \min(\beta_1, \beta_2)$. If, in addition, $\alpha_1, \beta_1 > r - 1$, then

$$\|\phi\|_{H^{r,s}_{\chi^{(\alpha,\beta)}}} \leqslant c\big(M^{r} + N^s\big)\|\phi\|_{\chi^{(\theta_1,\theta_2)}},$$

where $\theta_1 = \min(\alpha_1 - r, \alpha_2)$ and $\theta_2 = \min(\beta_1 - r, \beta_2)$.

Proof. Let $\phi_l(x_1)$ be the same as in (3.1). By Lemma 2.2, for any integers $0 \leqslant \mu \leqslant r$,

$$\begin{aligned}\big\|\partial^\mu_{x_1}\phi\big\|^2_{\chi^{(\alpha_1,\beta_1)}} &= 2\pi \sum_{|l|\leqslant N} \big\|\partial^\mu_{x_1}\phi_l\big\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} \leqslant cM^{4\mu} \sum_{|l|\leqslant N} \|\phi_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} \\ &\leqslant cM^{4\mu}\|\phi\|^2_{\chi^{(\eta_1,\eta_2)}}.\end{aligned}$$

Similarly, for any integers $0 \leqslant \nu \leqslant s$, $\|\partial^\nu_{x_2}\phi\|^2_{\chi^{(\alpha_2,\beta_2)}} \leqslant cN^{2\nu}\|\phi\|^2_{\chi^{(\eta_1,\eta_2)}}$. The previous statements with space interpolation lead to the first result. We next prove the second result. Due to $\alpha_1, \beta_1 > r - 1$, we use Lemma 2.2 to obtain that for integer $\mu \geqslant 0$,

$$\big\|\partial^\mu_{x_1}\phi\big\|^2_{\chi^{(\alpha_1,\beta_1)}} \leqslant cM^{2\mu} \sum_{|l|\leqslant N} \|\phi_l\|^2_{\chi^{(\alpha_1-r,\beta_1-r)},\Lambda_1} \leqslant cM^{2\mu}\|\phi\|^2_{\chi^{(\theta_1,\theta_2)}}.$$

The rest part of the proof is clear. $\square$

We now turn to some orthogonal projections. To do this, we introduce the space

$$H^{r,s}_{\chi^{(\gamma,\delta)},A}(\Omega) = L^2\big(\Lambda_2; H^r_{\chi^{(\gamma,\delta)},A}(\Lambda_1)\big) \cap H^s\big(\Lambda_2; L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1)\big), \quad r, s \geqslant 0,$$

with the norm

$$\|v\|_{H^{r,s}_{\chi^{(\gamma,\delta)},A}} = \Big(\|v\|^2_{L^2(\Lambda_2;H^r_{\chi^{(\gamma,\delta)},A}(\Lambda_1))} + \|v\|^2_{H^s(\Lambda_2;L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1))}\Big)^{1/2}.$$

Denote by $H^{r,s}_{p,\chi^{(\gamma,\delta)},A}(\Omega)$ the subspace of $H^{r,s}_{\chi^{(\gamma,\delta)},A}(\Omega)$, consisting of functions whose derivatives of order up to $r-1$ has the period 2π for the variable x_2. Throughout this paper, all spaces with the subscript p have the similar meanings.

The orthogonal projection $P_{M,N,\gamma,\delta} : L^2_{p,\chi^{(\gamma,\delta)}}(\Omega) \to V_{M,N}$ is defined by

$$(P_{M,N,\gamma,\delta}v - v, \phi)_{\chi^{(\gamma,\delta)}} = 0, \quad \forall \phi \in V_{M,N}.$$

Theorem 3.3. *For any $v \in H^{r,s}_{p,\chi^{(\gamma,\delta)},A}(\Omega)$ and integers $r, s \geqslant 0$,*

$$\|P_{M,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c\big(M^{-r} + N^{-s}\big)\|v\|_{H^{r,s}_{\chi^{(\gamma,\delta)},A}}.$$

Proof. Let

$$v_l(x_1) = \frac{1}{2\pi}\int_{\Lambda_2} v(x_1, x_2)e^{-ilx_2}\,dx_2, \tag{3.2}$$

and $P_{M,\gamma,\delta,\Lambda_1}$ be the $L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1)$-orthogonal projection as defined in Section 2. Clearly,

$$P_{M,N,\gamma,\delta}v(x) = \sum_{|l|\leqslant N}\big(P_{M,\gamma,\delta,\Lambda_1}v_l(x_1)\big)e^{ilx_2}.$$

By Lemma 2.3 and the result on the Fourier approximation,

$$\begin{aligned}
&\|P_{M,N,\gamma,\delta}v - v\|^2_{\chi^{(\gamma,\delta)}} \\
&\quad = 2\pi\sum_{|l|\leqslant N}\|P_{M,\gamma,\delta,\Lambda_1}v_l - v_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1} + 2\pi\sum_{|l|>N}\|v_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1} \\
&\quad \leqslant cM^{-2r}\|v\|^2_{L^2(\Lambda_2;H^r_{\chi^{(\gamma,\delta)},A}(\Lambda_1))} + cN^{-2s}\|v\|^2_{H^s(\Lambda_2;L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1))}. \qquad \square
\end{aligned}$$

In many cases, the coefficients of differential equations degenerate in different ways. Thus we need to consider the related orthogonal projections in certain non-uniformly weighted Sobolev spaces. For this purpose, let $H^0_{\alpha,\beta,\gamma,\delta}(\Omega) = L^2_{\chi^{(\gamma,\delta)}}(\Omega)$, and define

$$H^1_{\alpha,\beta,\gamma,\delta}(\Omega) = \big\{v \mid v \text{ is measurable on } \Omega \text{ and } \|v\|_{1,\alpha,\beta,\gamma,\delta} < \infty\big\},$$

equipped with the following semi-norm and norm:

$$\begin{aligned}
|v|_{1,\alpha,\beta,\gamma,\delta} &= \big(\|\partial_{x_1}v\|^2_{\chi^{(\alpha_1,\beta_1)}} + \|\partial_{x_2}v\|^2_{\chi^{(\alpha_2,\beta_2)}}\big)^{1/2}, \\
\|v\|_{1,\alpha,\beta,\gamma,\delta} &= \big(|v|^2_{1,\alpha,\beta,\gamma,\delta} + \|v\|^2_{\chi^{(\gamma,\delta)}}\big)^{1/2}.
\end{aligned}$$

For $0 < \mu < 1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Omega)$ is defined by space interpolation. In particular, ${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ and $v(-1, x_2) = 0$ for $x_2 \in \Lambda_2\}$, and $H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ and $v(-1, x_2) = v(1, x_2) = 0$ for $x_2 \in \Lambda_2\}$. The space $H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ is defined as before.

Now, let

$$a_{\alpha,\beta,\gamma,\delta}(u,v)=(\partial_{x_1}u,\partial_{x_1}v)_{\chi^{(\alpha_1,\beta_1)}}+(\partial_{x_2}u,\partial_{x_2}v)_{\chi^{(\alpha_2,\beta_2)}}+(u,v)_{\chi^{(\gamma,\delta)}}.$$

The orthogonal projection $P^1_{M,N,\alpha,\beta,\gamma,\delta}: H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)\to V_{M,N}$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}\big(P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v,\phi\big)=0,\quad \forall\phi\in V_{M,N}.$$

The orthogonal projection ${}_0P^1_{M,N,\alpha,\beta,\gamma,\delta}: {}_0H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)\to {}_0V_{M,N}$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}\big({}_0P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v,\phi\big)=0,\quad \forall\phi\in {}_0V_{M,N}.$$

The orthogonal projection $\tilde P^{1,0}_{M,N,\alpha,\beta,\gamma,\delta}: H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega)\cap H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)\to V^0_{M,N}$ is defined by

$$\big(\partial_{x_1}\big(\tilde P^{1,0}_{M,N,\alpha,\beta,\gamma,\delta}v-v\big),\partial_{x_1}\phi\big)_{\chi^{(\alpha_1,\beta_1)}}+\big(\partial_{x_2}\big(\tilde P^{1,0}_{M,N,\alpha,\beta,\gamma,\delta}v-v\big),\partial_{x_2}\phi\big)_{\chi^{(\alpha_2,\beta_2)}}=0,$$
$$\forall\phi\in V^0_{M,N}.$$

For description of approximation results, we define the non-isotropic space

$$M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}(\Omega)=L^2\big(\Lambda_2;H^r_{\chi^{(\alpha_1,\beta_1)},*}(\Lambda_1)\big)\cap H^1\big(\Lambda_2;H^{r+\sigma-2}_{\chi^{(\alpha_1,\beta_1)},*}(\Lambda_1)\big)$$
$$\cap H^{s-1}\big(\Lambda_2;H^1_{\alpha_1,\beta_1,\gamma,\delta}(\Lambda_1)\big)\cap H^s\big(\Lambda_2;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1)\big),$$

where $r,s\geqslant 1$, $\sigma=1$ or 2. Its norm $\|v\|_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}}$ is defined in the usual way. The meaning of $M^{r,s,\sigma}_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ is clear. One of the main results in this section is stated below.

Theorem 3.4. *If*

$$\alpha_1\leqslant\gamma+\sigma,\qquad \beta_1\leqslant\delta+\sigma,\quad \gamma\leqslant\alpha_2,\ \delta\leqslant\beta_2,\ \sigma=1\text{ or }2, \tag{3.3}$$

then for any $v\in M^{r,s,\sigma}_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ *and integers* $r,s\geqslant 1$,

$$\big\|P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\big\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\big(M^{1-r}+N^{1-s}\big)\|v\|_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}}. \tag{3.4}$$

If, in addition, $\sigma=1$, *then for all* $0\leqslant\mu\leqslant 1$,

$$\big\|P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\big\|_{\mu,\alpha,\beta,\gamma,\delta}$$
$$\leqslant c\big(M^{1-r}+N^{1-s}\big)^{\mu}\big(\big(M^{-r}+N^{-s}\big)\big(1+M^{-1}N\big)\big)^{1-\mu}\|v\|_{M^{r,s,1}_{\alpha,\beta,\gamma,\delta}}. \tag{3.5}$$

Proof. Let $v_l(x_1)$ be the same as in (3.2), and $P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}$ be the $H^1_{\alpha_1,\beta_1,\gamma,\delta}(\Lambda_1)$-orthogonal projection as defined in Section 2. Set

$$v^*(x)=\sum_{|l|\leqslant N}\big(P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}v_l(x_1)\big)e^{ilx_2}\in V_{M,N}.$$

By the projection theorem, (3.3), Lemma 2.4 and the result on the Fourier approximation, we obtain that for $\sigma=2$, $r\geqslant 1$ (or $\sigma=1$, $r\geqslant 2$) and $s\geqslant 1$,

$$\big\|P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\big\|^2_{1,\alpha,\beta,\gamma,\delta}$$
$$=\inf_{\phi\in V_{M,N}}\|\phi-v\|^2_{1,\alpha,\beta,\gamma,\delta}\leqslant\|v^*-v\|^2_{1,\alpha,\beta,\gamma,\delta}$$

$$\begin{aligned}
&\leqslant c \sum_{|l|\leqslant N} \left(\left\| P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} v_l - v_l \right\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \left\| P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} v_l - v_l \right\|^2_{\chi^{(\gamma,\delta)},\Lambda_1} \right) \\
&\quad + c \sum_{|l|>N} \left(\|v_l\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \|v_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} \right) \\
&\leqslant c M^{2-2r} \left(\|v\|^2_{L^2(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)},*}(\Lambda_1))} + \|v\|^2_{H^1(\Lambda_2; H^{r+\sigma-2}_{\chi^{(\alpha_1,\beta_1)},*}(\Lambda_1))} \right) \\
&\quad + c N^{2-2s} \left(\|v\|^2_{H^{s-1}(\Lambda_2; H^1_{\alpha_1,\beta_1,\gamma,\delta}(\Lambda_1))} + \|v\|^2_{H^s(\Lambda_2; L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1))} \right) \\
&\leqslant c \left(M^{1-r} + N^{1-s} \right)^2 \|v\|^2_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}}.
\end{aligned}$$

If $r=1$, $\sigma=1$ and $s \geqslant 1$, then we obtain from the projection theorem that

$$\left\| P^1_{M,N,\alpha,\beta,\gamma,\delta} v - v \right\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant \|v\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant c \|v\|^2_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}}.$$

We now use an duality argument to prove (3.5) with $\sigma = 1$ and $\mu = 0$. Let

$$\begin{aligned}
&a_{l,\alpha,\beta,\gamma,\delta}(u,v) = (\partial_{x_1} u, \partial_{x_1} v)_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} + l^2 (u,v)_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} + (u,v)_{\chi^{(\gamma,\delta)},\Lambda_1}, \\
&P^1_{M,N,\alpha,\beta,\gamma,\delta} v(x) = \sum_{|l|\leqslant N} v^*_l(x_1) e^{ilx_2}.
\end{aligned}$$

It can be checked that $v^*_l \in \mathcal{P}_M(\Lambda_1)$ and $a_{l,\alpha,\beta,\gamma,\delta}(v^*_l - v_l, \phi) = 0$, $\forall \phi \in \mathcal{P}_M(\Lambda_1)$. Thus we use the above and (3.3) to verify that for any $\phi \in \mathcal{P}_M(\Lambda_1)$,

$$\begin{aligned}
&\left\| v^*_l - v_l \right\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \left\| v^*_l - v_l \right\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} \\
&\quad = a_{l,\alpha,\beta,\gamma,\delta}\left(v^*_l - v_l, v^*_l - v_l \right) \\
&\quad = a_{l,\alpha,\beta,\gamma,\delta}\left(v^*_l - v_l, \phi - v_l \right) \leqslant c \left(\|\phi - v_l\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \|\phi - v_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1} \right).
\end{aligned}$$

Taking $\phi = P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} v_l$ and using Lemma 2.4 and (3.3), we deduce that

$$\begin{aligned}
&\left\| v^*_l - v_l \right\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \left\| v^*_l - v_l \right\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} \\
&\quad \leqslant c M^{2-2r} \left(\|v_l\|^2_{r,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1} + l^2 \|v_l\|^2_{r+\sigma-2,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1} \right). \qquad (3.6)
\end{aligned}$$

Next, let $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1)$ and consider the following auxiliary problem:

$$a_{l,\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)},\Lambda_1}, \quad \forall z \in H^1_{\alpha_1,\beta_1,\gamma,\delta}(\Lambda_1). \qquad (3.7)$$

Taking $z = w$ in (3.7), we verify that

$$\|w\|_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1} + l^2 \|w\|_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} \leqslant c \|g\|_{\chi^{(\gamma,\delta)},\Lambda_1}. \qquad (3.8)$$

Now let $w(x_1)$ vary in $\mathcal{D}(\Lambda_1)$, and so by (3.7), in the sense of distributions,

$$-\partial_{x_1}\left(\partial_{x_1} w(x_1) \chi^{(\alpha_1,\beta_1)}(x_1) \right) = \left(g(x_1) - w(x_1) \right) \chi^{(\gamma,\delta)}(x_1) - l^2 w(x_1) \chi^{(\alpha_2,\beta_2)}(x_1). \qquad (3.9)$$

As in the proof of Theorem 2.5 of [4], we can show that $\partial_{x_1} w(1) \chi^{(\alpha_1,\beta_1)}(1) = \partial_{x_1} w(-1) \times \chi^{(\alpha_1,\beta_1)}(-1) = 0$. Moreover, we obtain from (3.9) that

$$\begin{aligned}
-\partial^2_{x_1} w(x_1) &= -\left((\alpha_1 + \beta_1) x_1 + (\alpha_1 - \beta_1) \right) \left(1 - x_1^2 \right)^{-1} \partial_{x_1} w(x_1) \\
&\quad + \left(g(x_1) - w(x_1) \right) \chi^{(\gamma-\alpha_1,\delta-\beta_1)}(x_1) - l^2 w(x_1) \chi^{(\alpha_2-\alpha_1,\beta_2-\beta_1)}(x_1). \qquad (3.10)
\end{aligned}$$

We now estimate $\|\partial_{x_1}^2 w(1-x_1^2)^{1/2}\|_{\chi^{(\alpha_1,\beta_1)},\Lambda_1}$. Let $\Lambda_1^{(1)}=[0,1)$ and $\Lambda_1^{(2)}=(-1,0)$. It can be verified that $\|\partial_{x_1}^2 w(x_1)(1-x_1^2)^{1/2}\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1}\leqslant D_1+D_2$ where

$$D_1=D_1\big(\Lambda_1^{(1)}\big)+D_1\big(\Lambda_1^{(2)}\big),$$
$$D_1\big(\Lambda_1^{(j)}\big)=8\big(\alpha_1^2+\beta_1^2\big)\int\limits_{\Lambda_1^{(j)}}\big(\partial_{x_1}w(x_1)\big)^2\chi^{(\alpha_1-1,\beta_1-1)}(x_1)\,dx_1,\quad j=1,2,$$
$$D_2=2\int\limits_{\Lambda_1}\big(g(x_1)-w(x_1)\big)^2\chi^{(2\gamma-\alpha_1+1,2\delta-\beta_1+1)}(x_1)\,dx_1$$
$$+2l^4\int\limits_{\Lambda_1}w^2(x_1)\chi^{(2\alpha_2-\alpha_1+1,2\beta_2-\beta_1+1)}(x_1)\,dx_1.$$

Due to (3.3) and $\sigma=1$, $\alpha_1\leqslant\min(\gamma+1,\alpha_2+1)$ and $\beta_1\leqslant\min(\delta+1,\beta_2+1)$. So by (3.8),

$$|D_2|\leqslant c\big(\|g-w\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}+l^4\|w\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big)\leqslant c\|g\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}.$$

Next, integrating (3.9) and inserting the result into $D_1(\Lambda_1^{(1)})$, we obtain that for $\sigma=1$,

$$D_1\big(\Lambda_1^{(1)}\big)$$
$$=c\int\limits_{\Lambda_1^{(1)}}\chi^{(-\alpha_1-1,-\beta_1-1)}(x_1)\left(\int_{x_1}^1\Big(\big(g(y)-w(y)\big)\chi^{(\gamma,\delta)}(y)-l^2w(y)\chi^{(\alpha_2,\beta_2)}(y)\Big)\,dy\right)^2dx_1$$
$$\leqslant c\int_0^1(1-x_1)^{-\gamma}\left(\frac{1}{1-x_1}\int_{x_1}^1\Big(\big(g(y)-w(y)\big)\chi^{(\gamma,\delta)}(y)-l^2w(y)\chi^{(\alpha_2,\beta_2)}(y)\Big)\,dy\right)^2dx_1.$$

By the Hardy inequality, for any measurable function $\Phi(x_1)$, real numbers $a\leqslant b$ and $d<1$,

$$\int_a^b\left(\frac{1}{b-x_1}\int_{x_1}^b\Phi(y)\,dy\right)^2(b-x_1)^d\,dx_1\leqslant\frac{4}{1-d}\int_a^b\Phi^2(x_1)(b-x_1)^d\,dx_1.\tag{3.11}$$

Letting $a=-1$, $b=1$ and $d=-\gamma$ in (3.11), we use (3.3) and (3.8) to derive that

$$D_1\big(\Lambda_1^{(1)}\big)\leqslant c\int\limits_{\Lambda_1^{(1)}}\big(g(x_1)-w(x_1)\big)^2\chi^{(\gamma,2\delta)}(x_1)\,dx_1+l^4\int\limits_{\Lambda_1^{(1)}}w^2(x_1)\chi^{(2\alpha_2-\gamma,2\beta_2)}(x_1)\,dx_1$$
$$\leqslant c\|g\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}.$$

A similar estimate is valid on $\Lambda_1^{(2)}$ for $\sigma=1$. A combination of the previous statements leads to that for $\gamma\leqslant\alpha_2$, $\delta\leqslant\beta_2$ and $\sigma=1$,

$$|w|_{2,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}=\big\|\partial_{x_1}^2w\big(1-x_1^2\big)^{1/2}\big\|_{\chi^{(\alpha_1,\beta_1)},\Lambda_1}\leqslant c\|g\|_{\chi^{(\gamma,\delta)},\Lambda_1}.\tag{3.12}$$

Now, we take $z=v_l^*-v_l$ in (3.7), and use (3.3), (3.12) and the second result of Lemma 2.4 to obtain that for $\sigma=1$ and $|l|\leqslant N$,

$$
\begin{aligned}
&\left|\left(v_l^*-v_l,g\right)_{\chi^{(\gamma,\delta)},\Lambda_1}\right| \\
&\quad=\left|a_{l,\alpha,\beta,\gamma,\delta}\left(v_l^*-v_l,w\right)\right|=\left|a_{l,\alpha,\beta,\gamma,\delta}\left(v_l^*-v_l,P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}w-w\right)\right| \\
&\quad\leqslant\left(\left\|v_l^*-v_l\right\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}+l^2\left\|v_l^*-v_l\right\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\right)^{1/2} \\
&\qquad\times\left(\left\|P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}w-w\right\|^2_{1,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}+l^2\left\|P^1_{M,\alpha_1,\beta_1,\gamma,\delta,\Lambda_1}w-w\right\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}\right)^{1/2} \\
&\quad\leqslant cM^{-r}\left(|v_l|^2_{r,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}+l^2|v_l|^2_{r-1,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}\right)^{1/2} \\
&\qquad\times\left(|w|^2_{2,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}+l^2M^{-2}|w|^2_{2,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}\right)^{1/2} \\
&\quad\leqslant cM^{-r}\left(1+M^{-1}N\right)\left(|v_l|^2_{r,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}+l^2|v_l|^2_{r-1,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}\right)^{1/2}\|g\|_{\chi^{(\gamma,\delta)},\Lambda_1}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|v_l^*-v_l\|_{\chi^{(\gamma,\delta)},\Lambda_1} \\
&\quad=\sup_{\substack{g\in L^2_{\chi^{(\gamma,\delta)}}(\Lambda_1)\\ g\neq 0}}\frac{|(v_l^*-v_l,g)_{\chi^{(\gamma,\delta)},\Lambda_1}|}{\|g\|_{\chi^{(\gamma,\delta)},\Lambda_1}} \\
&\quad\leqslant cM^{-r}\left(1+M^{-1}N\right)\left(\|v_l\|^2_{r,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}+l^2\|v_l\|^2_{r-1,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}\right)^{1/2}.
\end{aligned}
\tag{3.13}
$$

Therefore,

$$
\begin{aligned}
&\left\|P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\right\|^2_{\chi^{(\gamma,\delta)}} \\
&\quad=2\pi\left(\sum_{|l|\leqslant N}\left\|v_l^*-v_l\right\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}+\sum_{|l|>N}\|v_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}\right) \\
&\quad\leqslant cM^{-2r}\left(1+M^{-1}N\right)^2\sum_{|l|\leqslant N}\left(\|v_l\|^2_{r,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}+l^2\|v_l\|^2_{r-1,\chi^{(\alpha_1,\beta_1)},*,\Lambda_1}\right) \\
&\qquad+cN^{-2s}\sum_{|l|>N}l^{2s}\|v_l\|^2_{\chi^{(\gamma,\delta)},\Lambda_1}\leqslant c\left(M^{-r}+N^{-s}\right)^2\left(1+M^{-1}N\right)^2\|v\|^2_{M^{r,s,1}_{\alpha,\beta,\gamma,\delta}}.
\end{aligned}
$$

The result (3.5) with $0\leqslant\mu\leqslant1$ comes from the previous results and space interpolation. □

By using Lemmas 2.5 and 2.6, we can prove the following results.

Theorem 3.5. *Let $\gamma\leqslant\alpha_2$ and $\delta\leqslant\beta_2$. If $\alpha_1\leqslant\gamma+1$, $\beta_1\leqslant\delta+2$, $0<\alpha_1<1$, $\beta_1<1$, or $\alpha_1\leqslant\gamma+2$, $\beta_1\leqslant0$, $\delta\geqslant0$, then for any $v\in{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Omega)\cap M^{r,s,2}_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ and integers $r,s\geqslant1$,*

$$
\left\|{}_0P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\right\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\left(M^{1-r}+N^{1-s}\right)\|v\|_{M^{r,s,2}_{\alpha,\beta,\gamma,\delta}}.
$$

If, in addition, $\beta_1<\delta+1$, then for all $0\leqslant\mu\leqslant1$,

$$
\begin{aligned}
&\left\|{}_0P^1_{M,N,\alpha,\beta,\gamma,\delta}v-v\right\|_{\mu,\alpha,\beta,\gamma,\delta} \\
&\quad\leqslant c\left(M^{1-r}+N^{1-s}\right)^{\mu}\left(\left(M^{-r}+N^{-s}\right)\left(1+M^{-1}N\right)\right)^{1-\mu}\|v\|_{M^{r,s,1}_{\alpha,\beta,\gamma,\delta}}.
\end{aligned}
$$

Theorem 3.6. *Let $\gamma \leqslant \alpha_2$ and $\delta \leqslant \beta_2$. If $-1 < \alpha_1, \beta_1 < 1$, then for any $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap M^{r,s,2}_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ and integers $r, s \geqslant 1$,*

$$\big\| \tilde{P}^{1,0}_{M,N,\alpha,\beta,\gamma,\delta} v - v \big\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c\big(M^{1-r} + N^{1-s}\big)\|v\|_{M^{r,s,2}_{\alpha,\beta,\gamma,\delta}}.$$

If, in addition, $\alpha_1, \beta_1 \leqslant 0$ or $\alpha_1, \beta_1 > 0$, then for all $0 \leqslant \mu \leqslant 1$,

$$\begin{aligned}&\big\| \tilde{P}^{1,0}_{M,N,\alpha,\beta} v - v \big\|_{\mu,\alpha,\beta,\gamma,\delta} \\ &\quad\leqslant c\big(M^{1-r} + N^{1-s}\big)^{\mu}\big(\big(M^{-r} + N^{-s}\big)\big(1 + M^{-1}N\big)\big)^{1-\mu}\|v\|_{M^{r,s,1}_{\alpha,\beta,\gamma,\delta}}.\end{aligned}$$

We now consider another orthogonal projection, which will be used for numerical solutions of semi-periodic problems on unbounded domains. Let $\nu = (\nu_0, \nu_1, \nu_2)$, $\nu_i > 0$, and

$$\hat{a}^{\nu}_{\alpha,\beta,0,0}(u,v) = \nu_1\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}\partial_{x_1}u, \partial_{x_1}\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}v\big)\big) + \nu_2\big(\chi^{(\alpha_2,\beta_2)}\partial_{x_2}u, \partial_{x_2}v\big) + \nu_0(u,v).$$

Lemma 3.1. *Let $\alpha_2, \beta_2 > -1$. If $\alpha_1 = \beta_1 = 2$, $\nu_0 > 2\nu_1$, or $\alpha_1 = 2$, $\beta_1 = 0$, $\nu_0 > \frac{1}{2}\nu_1$, then for any $u, v \in H^1_{p,\alpha,\beta,0,0}(\Omega)$,*

$$\hat{a}^{\nu}_{\alpha,\beta,0,0}(v,v) \geqslant c(\nu)\|v\|^2_{1,\alpha,\beta,0,0}, \tag{3.14}$$

$$\hat{a}^{\nu}_{\alpha,\beta,0,0}(u,v) \leqslant c\|u\|_{1,\alpha,\beta,0,0}\|v\|_{1,\alpha,\beta,0,0}, \tag{3.15}$$

where $c(\nu) = \min(\nu_0 - 2\nu_1, \nu_1, \nu_2)$ for $\nu_0 > 2\nu_1$, and $c(\nu) = \min(\nu_0 - \frac{1}{2}\nu_1, \nu_1, \nu_2)$ for $\nu_0 > \frac{1}{2}\nu_1$.

Proof. Let $v_l(x_1)$ be the same as in (3.2). If $\alpha_1 = \beta_1 = 2$, then integrating by parts yields that

$$\begin{aligned}&\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}\partial_{x_1}v_l, \partial_{x_1}\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}v_l\big)\big)_{\Lambda_1} \\ &\quad= \|\partial_{x_1}v_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} - \int_{\Lambda_1} x_1\big(1 - x_1^2\big)\partial_{x_1}\big(v_l^2(x_1)\big)\,dx_1 \\ &\quad= \|\partial_{x_1}v_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} + \|v_l\|^2_{\Lambda_1} - 3\int_{\Lambda_1} x_1^2 v_l^2(x_1)\,dx_1 \\ &\quad\geqslant \|\partial_{x_1}v_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} - 2\|v_l\|^2_{\Lambda_1}.\end{aligned}$$

Thus

$$\begin{aligned}&\hat{a}^{\nu}_{\alpha,\beta,0,0}(v,v) \\ &\quad= 2\pi\sum_{|l|=0}^{\infty}\big(\nu_1\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}\partial_{x_1}v_l, \partial_{x_1}\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}v_l\big)\big)_{\Lambda_1} + \nu_2 l^2\|v_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} + \nu_0\|v_l\|^2_{\Lambda_1}\big) \\ &\quad\geqslant 2\pi\sum_{|l|=0}^{\infty}\big(\nu_1\|\partial_{x_1}v_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} + \nu_2 l^2\|v_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} + (\nu_0 - 2\nu_1)\|v_l\|^2_{\Lambda_1}\big) \\ &\quad\geqslant \min(\nu_0 - 2\nu_1, \nu_1, \nu_2)\|v\|^2_{1,\alpha,\beta,0,0}.\end{aligned} \tag{3.16}$$

The above implies the result (3.14) with $\nu_0 > 2\nu_1$. If $\alpha_1 = 2$ and $\beta_1 = 0$, then

$$\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}\partial_{x_1} v_l, \partial_{x_1}\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})} v_l\big)\big)_{\Lambda_1} = \|\partial_{x_1} v_l\|^2_{\chi^{(\alpha_1,\beta_1)},\Lambda_1} - \frac{1}{2}\|v_l\|^2_{\Lambda_1} + v_l^2(-1).$$

This fact leads to (3.14) with $\nu_0 > \frac{1}{2}\nu_1$. The proof of (3.15) is clear. □

The orthogonal projection $\hat{P}^1_{M,N,\alpha,\beta,0,0} : H^1_{p,\alpha,\beta,0,0}(\Omega) \to V_{M,N}$ is defined by

$$\hat{a}^{\nu}_{\alpha,\beta,0,0}\big(\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v, \phi\big) = 0, \quad \forall \phi \in V_{M,N}.$$

The orthogonal projection ${}_0\hat{P}^1_{M,N,\alpha,\beta,0,0} : {}_0H^1_{p,\alpha,\beta,0,0}(\Omega) \to {}_0V_{M,N}$ is defined by

$$\hat{a}^{\nu}_{\alpha,\beta,0,0}\big({}_0\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v, \phi\big) = 0, \quad \forall \phi \in {}_0V_{M,N}.$$

For description of approximation result, we introduce the space

$$\hat{M}^{r,s}_{\alpha,\beta,,0,0}(\Omega) = L^2\Big(\Lambda_2; H^r_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1)\Big) \cap H^1\Big(\Lambda_2; H^{r-1}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1)\Big)$$
$$\cap H^{s-1}\big(\Lambda_2; H^1_{\alpha_1,\beta_1,0,0}(\Lambda_1)\big) \cap H^s\big(\Lambda_2; L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1)\big).$$

Its norm $\|v\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}}$ is defined in the usual way.

Theorem 3.7. *Let $\alpha_2, \beta_2 \geqslant 0$, and the conditions of Lemma 3.1 hold. Then for any $v \in \hat{M}^{r,s}_{p,\alpha,\beta,0,0}(\Omega)$ and integers $r, s \geqslant 1$,*

$$\big\|\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v\big\|_{1,\alpha,\beta,0,0} \leqslant c\big(M^{1-r} + N^{1-s}\big)\|v\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}}.$$

Proof. Let $v_l(x_1)$ be the same as before, and

$$v^*(x) = \sum_{|l|\leqslant N}\big(P^1_{M,\frac{\alpha_1}{2},\frac{\beta_1}{2},0,0,\Lambda_1} v_l(x_1)\big)e^{ilx_2}.$$

Clearly, $v^* \in V_{M,N}$. So by Lemma 3.1 and the definition of $\hat{P}^1_{M,N,\alpha,\beta,0,0}$,

$$\begin{aligned}
&c(\nu)\big\|\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v\big\|^2_{1,\alpha,\beta,0,0} \\
&\quad\leqslant \hat{a}^{\nu}_{\alpha,\beta,0,0}\big(\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v, \hat{P}^1_{M,N,\alpha,\beta,0,0} v - v\big) \\
&\quad= \hat{a}^{\nu}_{\alpha,\beta,0,0}\big(\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v, \phi - v\big) \leqslant c\big\|\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v\big\|_{1,\alpha,\beta,0,0}\|\phi - v\|_{1,\alpha,\beta,0,0}.
\end{aligned}$$

Taking $\phi = v^*$ and using Lemma 2.4, we verify that for $r \geqslant 2$ and $s \geqslant 1$,

$$\begin{aligned}
&\big\|\hat{P}^1_{M,N,\alpha,\beta,0,0} v - v\big\|^2_{1,\alpha,\beta,0,0} \\
&\quad\leqslant c\Bigg(\sum_{|l|\leqslant N}\big\|P^1_{M,\frac{\alpha_1}{2},\frac{\beta_1}{2},0,0,\Lambda_1} v_l - v_l\big\|^2_{1,\frac{\alpha_1}{2},\frac{\beta_1}{2},0,0,\Lambda_1} + l^2\big\|P^1_{M,\frac{\alpha_1}{2},\frac{\beta_1}{2},0,0,\Lambda_1} v_l - v_l\big\|^2_{\Lambda_1}\Bigg) \\
&\qquad + c\sum_{|l|>N}\big(\|v_l\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|v_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big) \\
&\quad\leqslant cM^{2-2r}\Big(\|v\|^2_{L^2(\Lambda_2; H^r_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1))} + \|v\|^2_{H^1(\Lambda_2; H^{r-1}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1))}\Big) \\
&\qquad + cN^{2-2s}\Big(\|v\|^2_{H^{s-1}(\Lambda_2; H^1_{\alpha_1,\beta_1,0,0}(\Lambda_1))} + \|v\|^2_{H^s(\Lambda_2; L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_1))}\Big). \qquad \square
\end{aligned}$$

In numerical analysis of the mixed Fourier–Jacobi spectral methods for nonlinear problems, we have to estimate the norm $\|\hat{P}^1_{M,N,\alpha,\beta,0,0}v\|_{L^\infty}$.

Theorem 3.8. *If $M = O(N)$, and the conditions of Theorem 3.7 hold, then for any $v \in L^2_p(\Lambda_2; H^{1+d}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1) \cap H^d(\Lambda_1))$ and $d > 1$,*

$$\|\hat{P}^1_{M,N,\alpha,\beta,0,0}v\|_{L^\infty} \leqslant c\|v\|_{L^2(\Lambda_2;H^{1+d}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1)\cap H^d(\Lambda_1))}.$$

Proof. Let

$$\hat{a}^\nu_{l,\alpha,\beta,0,0}(u,v) = \nu_1\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}\partial_{x_1}u, \partial_{x_1}\big(\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})}v\big)\big)_{\Lambda_1} + \nu_2 l^2(u,v)_{\chi^{(\alpha_2,\beta_2)},\Lambda_1} + \nu_0(u,v)_{\Lambda_1},$$
$$\hat{P}^1_{M,N,\alpha,\beta,0,0}v(x) = \sum_{|l|\leqslant N} v^*_l(x_1)e^{ilx_2} \in \mathcal{P}_M(\Lambda_1).$$

By an argument as in the proof of Lemma 3.1, we assert that for certain $c(\nu) > 0$,

$$\hat{a}^\nu_{l,\alpha,\beta,0,0}(w,w) \geqslant c(\nu)\big(\|w\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|w\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big),$$
$$\big|\hat{a}^\nu_{l,\alpha,\beta,0,0}(w,z)\big|$$
$$\leqslant c\big(\|w\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|w\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big)^{1/2}\big(\|z\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|z\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big)^{1/2}.$$

Moreover, $\hat{a}^\nu_{l,\alpha,\beta,0,0}(v_l - v^*_l, \phi) = 0$ for all $\phi \in \mathcal{P}_M(\Lambda_1)$. Hence

$$\|v_l - v^*_l\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|v_l - v^*_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}$$
$$\leqslant c\hat{a}^\nu_{l,\alpha,\beta,0,0}\big(v_l - v^*_l, v_l - v^*_l\big) = c\hat{a}^\nu_{l,\alpha,\beta,0,0}\big(v_l - v^*_l, v_l - \phi\big)$$
$$\leqslant c\big(\|\phi - v_l\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|\phi - v_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}\big).$$

By taking $\phi = P^1_{M,\frac{\alpha_1}{2},\frac{\beta_1}{2},0,0,\Lambda_1}v_l$ and using Lemma 2.4, we deduce that for $|l| \leqslant N$,

$$\|v_l - v^*_l\|^2_{1,\alpha_1,\beta_1,0,0,\Lambda_1} + l^2\|v_l - v^*_l\|^2_{\chi^{(\alpha_2,\beta_2)},\Lambda_1}$$
$$\leqslant cM^{2-2r}\big(1 + N^2M^{-2}\big)\|v_l\|^2_{r,\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*,\Lambda_1}. \tag{3.17}$$

We now use (3.17) to derive the desired result. Due to $d > 1$ and the imbedding theorem,

$$\|v^*_l\|_{L^\infty,\Lambda_1} \leqslant \|v_l\|_{L^\infty,\Lambda_1} + c\|v^*_l - v_l\|_{\frac{d}{2},\Lambda_1}$$
$$\leqslant \|v_l\|_{L^\infty,\Lambda_1} + c\big(\|v^*_l - P_{M,0,0,\Lambda_1}v_l\|_{\frac{d}{2},\Lambda_1} + \|P_{M,0,0,\Lambda_1}v_l - v_l\|_{\frac{d}{2},\Lambda_1}\big).$$

Moreover, by Lemmas 2.2, 2.3, (3.17) and the fact $M = O(N)$,

$$\|v^*_l - P_{M,0,0,\Lambda_1}v_l\|_{\frac{d}{2},\Lambda_1} \leqslant cM^d\big(\|v^*_l - v_l\|_{\Lambda_1} + \|P_{M,0,0,\Lambda_1}v_l - v_l\|_{\Lambda_1}\big)$$
$$\leqslant c\Big(\|v_l\|_{1+d,\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*,\Lambda_1} + \|v_l\|_{d,\Lambda_1}\Big).$$

On the other hand (see [2]), $\|P_{M,0,0,\Lambda_1}v_l - v_l\|_{\frac{d}{2},\Lambda_1} \leqslant c\|v_l\|_{\frac{3d}{4},\Lambda_1}$. A combination of the previous estimates implies that

$$\|v^*_l\|_{L^\infty,\Lambda_1} \leqslant c\Big(\|v_l\|_{1+d,\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*,\Lambda_1} + \|v_l\|_{d,\Lambda_1}\Big).$$

Finally, the above with the result on the Fourier approximation lead to

$$\|\hat{P}_{M,N,\alpha,\beta,0,0}v\|^2_{L^\infty} \leqslant c\sum_{|l|\leqslant N}\|v_l^*\|^2_{L^\infty,\Lambda_1} \leqslant c\|v\|^2_{L^2(\Lambda_2;H^{1+d}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1)\cap H^d(\Lambda_1))}. \qquad \square$$

Remark 3.1. For $\alpha_2=\beta_2=0$, we can remove the condition $M=O(N)$ in Theorem 3.6. Indeed, in this case, $\|v_l^*-v_l\|_{\Lambda_1}\leqslant cM^{1-r}\|v_l\|_{r,\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*,\Lambda_1}$. So the conclusion follows as in the late part of the proof of Theorem 3.8.

By using Lemmas 2.5, 3.1 and the same argument as in the proof of Theorem 3.7, we can prove following result.

Theorem 3.9. *Let $\alpha_1=2$, $\beta_1=0$, $\alpha_2\geqslant 2$, $\beta_2=0$ and $\nu_0>\frac{1}{2}\nu_1$. Then for any $v\in_0 H^1_{\alpha,\beta,0,0}(\Omega)\cap \hat{M}^{r,s}_{p,\alpha,\beta,0,0}(\Omega)$ and integers $r,s\geqslant 1$,*

$$\big\|{}_0\hat{P}^1_{M,N,\alpha,\beta,0,0}v-v\big\|_{1,\alpha,\beta,0,0}\leqslant c\big(M^{1-r}+N^{1-s}\big)\|v\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}}.$$

4. Some applications

We first consider the following simple model problem,

$$-\partial_{x_1}\big(a_1(x)\partial_{x_1}U(x)\big)-\partial_{x_2}\big(a_2(x)\partial_{x_2}U(x)\big)+a_0(x)U(x)=f(x),\quad x\in\Omega. \tag{4.1}$$

Assume that $a_i(x)$ degenerate as $|x_1|\to 1$, and

$$\begin{aligned}&a_0(x)=\tilde{a}_0(x)\chi^{(\gamma,\delta)}(x),\qquad a_1(x)=\tilde{a}_1(x)\chi^{(\alpha_1,\beta_1)}(x),\\&a_2(x)=\tilde{a}_2(x)\chi^{(\alpha_2,\beta_2)}(x),\end{aligned} \tag{4.2}$$

with $\tilde{a}_i(x)\in L^\infty(\Omega)$, $\tilde{a}_i(x)\geqslant\tilde{a}^{(i)}_{\min}>0$, $i=0,1,2$. We look for solution of (4.1) such that

$$\lim_{|x_1|\to 1}a_1(x)U(x)\partial_{x_1}U(x)=0,\quad \forall x_2\in\Lambda_2. \tag{4.3}$$

For any $u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$, let

$$A_{\alpha,\beta,\gamma,\delta}(u,v)=(\tilde{a}_1\partial_{x_1}u,\partial_{x_1}v)_{\chi^{(\alpha_1,\beta_1)}}+(\tilde{a}_2\partial_{x_2}u,\partial_{x_2}v)_{\chi^{(\alpha_2,\beta_2)}}+(\tilde{a}_0u,v)_{\chi^{(\gamma,\delta)}}.$$

A weak formulation of (4.1) with (4.3) is to find $U\in H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ such that

$$A_{\alpha,\beta,\gamma,\delta}(U,v)=(f,v),\quad \forall v\in H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega). \tag{4.4}$$

By (4.2) and the Cauchy–Schwartz inequality, we find that for any $u,v\in H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)$,

$$\big|A_{\alpha,\beta,\gamma,\delta}(u,v)\big|\leqslant\big(\|\tilde{a}_1\|_{L^\infty}+\|\tilde{a}_2\|_{L^\infty}+\|\tilde{c}_0\|_{L^\infty}\big)\|u\|_{1,\alpha,\beta,\gamma,\delta}\|v\|_{1,\alpha,\beta,\gamma,\delta}, \tag{4.5}$$

$$\big|A_{\alpha,\beta,\gamma,\delta}(u,u)\big|\geqslant\min\big(\tilde{a}^{(0)}_{\min},\tilde{a}^{(1)}_{\min},\tilde{a}^{(2)}_{\min}\big)\|u\|^2_{1,\alpha,\beta,\gamma,\delta}. \tag{4.6}$$

Therefore, if $f\in L^2_{\chi^{(-\gamma,-\delta)}}(\Omega)$, then (4.4) has a unique solution such that $\|U\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\|f\|_{\chi^{(-\gamma,-\delta)}}$.

Now, let $u_{M,N}\in V_{M,N}$ be the approximation to U, satisfying

$$A_{\alpha,\beta,\gamma,\delta}(u_{M,N},\phi)=(f,\phi),\quad \forall\phi\in V_{M,N}. \tag{4.7}$$

Due to (4.2), (4.7) is also unisolvent and $\|u_{M,N}\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\|f\|_{\chi^{(-\gamma,-\delta)}}$.

Theorem 4.1. *Let U and $u_{M,N}$ be the solutions of* (4.1) *and* (4.7), *respectively. If* (3.3) *and* (4.2) *hold, and $U \in M^{r,s,\sigma}_{p,\alpha,\beta,\gamma,\delta}(\Omega)$ with integers $r, s \geqslant 1$ and $\sigma = 1, 2$, then*

$$\|U - u_{M,N}\|_{1,\alpha,\beta,\gamma,\delta} \leqslant d\bigl(M^{1-r} + N^{1-s}\bigr)\|U\|_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}},$$

where the constant d depends only on the norms $\|\tilde{a}_i\|_{L^\infty}$, $i = 0, 1, 2$.

Proof. Let $U_{M,N} = P^1_{M,N,\alpha,\beta,\gamma,\delta}U$. By (4.5)–(4.7) and Theorem 3.4,

$$\begin{aligned} c\|u_{M,N} - U_{M,N}\|^2_{1,\alpha,\beta,\gamma,\delta} &\leqslant A_{\alpha,\beta,\gamma,\delta}(u_{M,N} - U_{M,N}, u_{M,N} - U_{M,N}) \\ &= A_{\alpha,\beta,\gamma,\delta}(U - U_{M,N}, u_N - U_{M,N}) \\ &\leqslant d\bigl(N_1^{1-r} + N_2^{1-s}\bigr)\|U\|_{M^{r,s,\sigma}_{\alpha,\beta,\gamma,\delta}}\|u_{M,N} - U_{M,N}\|_{1,\alpha,\beta,\gamma,\delta}. \end{aligned}$$

Thus using Theorem 3.4 again completes the proof. □

We next consider the nonlinear Klein–Gordon equation, which plays an important role in quantum mechanics. Let $\Omega^* = \{y = (y_1, y_2) \mid -\infty < y_1 < +\infty,\ 0 \leqslant y_2 < 2\pi\}$. The problem is of the form

$$\begin{cases} \partial_t^2 V(y,t) - \Delta V(y,t) + V^3(y,t) = F(y,t), & (y,t) \in \Omega^* \times (0,T], \\ \partial_t V(y,0) = V_1(y,0), & y \in \Omega^*, \\ V(y,0) = V_1(y,0), & y \in \Omega^*. \end{cases} \tag{4.8}$$

Assume that all functions in (4.8) have the period 2π for the variable y_2, and $V(y,t)$ satisfies the following asymptotic boundary condition:

$$\lim_{|y_1|\to\infty} e^{-a|y_1|}\partial_{y_1}V(y_1, y_2, t) = 0, \quad a > 0,\ \forall (y_2, t) \in \Lambda_2 \times [0,T]. \tag{4.9}$$

Let $b > 0$, which should be chosen suitably based on the value of a. We make the following variable transformation:

$$y_1(x_1) = b\ln\frac{1+x_1}{1-x_1}, \qquad y_2(x_2) = x_2. \tag{4.10}$$

Furthermore,

$$\begin{aligned} &U(x_1,x_2,t) = V\bigl(y_1(x_1), y_2(x_2), t\bigr), && U_1(x_1,x_2) = V_1\bigl(y_1(x_1), y_2(x_2)\bigr), \\ &U_0(x_1,x_2) = V_0\bigl(y_1(x_1), y_2(x_2)\bigr), && f(x_1,x_2,t) = F\bigl(y_1(x_1), y_2(x_2), t\bigr). \end{aligned}$$

Then problem (4.8) with (4.9) becomes

$$\begin{cases} \partial_t^2 U(x,t) - \frac{1}{4b^2}(1-x_1^2)\partial_{x_1}((1-x_1^2)\partial_{x_1}U(x,t)) - \partial_{x_2}^2 U(x,t) + U^3(x,t) = f(x,t), \\ \quad (x,t) \in \Omega \times (0,T], \\ \lim_{|x_1|\to 1}(1-x_1^2)^{ab+1}\partial_{x_1}U(x,t) = 0, \quad (x_2,t) \in \Lambda_2 \times [0,T], \\ \partial_t U(x,0) = U_1(x), \quad U(x,0) = U_0(x), \quad x \in \Omega. \end{cases} \tag{4.11}$$

We now derive a weak form of (4.11). Multiplying the second term of (4.11) by $v \in H^1_{p,\alpha,\beta,0,0}(\Omega) \cap L^4_p(\Omega)$ and integrating the result by parts, we obtain the boundary term

$$B(U,v) = \frac{1}{4b^2}\lim_{x_1\to -1}\bigl(1-x_1^2\bigr)^2 v(x)\partial_{x_1}U(x) - \frac{1}{4b^2}\lim_{x_1\to 1}\bigl(1-x_1^2\bigr)^2 v(x)\partial_{x_1}U(x).$$

Since $v \in L^4(\Omega)$, we have $v = o((1-x_1^2)^{-1/4})$ as $|x_1| \to 1$. By the boundary condition (4.3), $B(U,v) = 0$ for $ab \leqslant \frac{3}{4}$. Next, let $\alpha_1 = \beta_1 = 2$, $\alpha_2 = \beta_2 = 0$, $\nu_0 > \frac{1}{2b^2}$, $\nu_1 = \frac{1}{4b^2}$ and $\nu_2 = 1$. Then the weak formulation of (4.11) is to find $U \in L^\infty(0,T; L^4_p(\Omega) \cap H^1_{p,\alpha,\beta,0,0}(\Omega)) \cap W^{1,\infty}(0,T; L^2_p(\Omega))$ such that

$$\begin{cases} (\partial_t^2 U(t) + U^3(t), v) + \hat{a}^{\nu}_{\alpha,\beta}(U(t), v) = (\nu_0 U(t) + f(t), v), \\ \quad \forall v \in L^4_p(\Omega) \cap H^1_{p,\alpha,\beta,0,0}(\Omega),\ t \in (0,T], \\ \partial_t U(0) = U_1, \quad U(0) = U_0. \end{cases} \tag{4.12}$$

Obviously the condition of Lemma 3.1 is fulfilled. So for suitable U_1, U_0 and f, (4.12) has a unique solution.

The mixed Fourier–Jacobi spectral scheme for (4.12) is to find $u_{M,N}(t) \in V_{M,N}$ such that

$$\begin{cases} (\partial_t^2 u_{M,N}(t) + u^3_{M,N}(t), \phi) + \hat{a}^{\nu}_{\alpha,\beta,0,0}(u_{M,N}(t), \phi) = (\nu_0 u_{M,N}(t) + f(t), \phi), \\ \quad \forall \phi \in V_{M,N},\ t \in (0,T], \\ \partial_t u_{M,N}(0) = u_{M,N,1} = P_{M,N,0,0} U_1, \\ u_{M,N}(0) = u_{M,N,0} = P_{M,N,0,0} U_0. \end{cases} \tag{4.13}$$

We now analyze the stability of scheme (4.13). Assume that the data $u_{M,N,0}$, $u_{M,N,1}$ and f are disturbed by $\tilde{u}_{M,N,0}$, $\tilde{u}_{M,N,1}$ and $\tilde{f}$, respectively, which induce the error of $u_{M,N}$, denoted by $\tilde{u}_{M,N}$. Then for all $\phi \in V_{M,N}$ and $t \in (0,T]$,

$$\left(\partial_t^2 \tilde{u}_{M,N}(t) + \tilde{u}^3_{M,N}(t) + G_0(t), \phi\right) + \hat{a}^{\nu}_{\alpha,\beta}\left(\tilde{u}_{M,N}(t), \phi\right) = \left(\nu_0 \tilde{u}_{M,N}(t) + \tilde{f}(t), \phi\right), \tag{4.14}$$

with

$$G_0(t) = 3\tilde{u}^2_{M,N}(t) u_{M,N}(t) + 3\tilde{u}_{M,N}(t) u^2_{M,N}(t),$$
$$\partial_t \tilde{u}_{M,N}(0) = \tilde{u}_{M,N,1}, \qquad \tilde{u}_{M,N}(0) = \tilde{u}_{M,N,0}.$$

Let

$$E(\tilde{u}_{M,N}, t) = \frac{1}{4}\left\|\tilde{u}_{M,N}(t)\right\|^2_{1,\alpha,\beta,0,0} + \frac{1}{2}\left\|\tilde{u}_{M,N}(t)\right\|^4_{L^4} + \left\|\partial_t \tilde{u}_{M,N}(t)\right\|^2.$$

We take $\phi = 2\partial_t \tilde{u}_{M,N}(t)$ in (4.14). Thanks to Lemma 3.1 and the fact that

$$\begin{aligned} \left|2\left(G_0(t), \partial_t \tilde{u}_{M,N}(t)\right)\right| &\leqslant \frac{1}{2}\left\|\tilde{u}_{M,N}(t)\right\|^4_{L^4} + \frac{1}{2}\left\|\tilde{u}_{M,N}(t)\right\|^2 \\ &\quad + 18\left(\left\|u_{M,N}(t)\right\|^2_\infty + \left\|u_{M,N}(t)\right\|^4_\infty\right)\left\|\partial_t \tilde{u}_{M,N}(t)\right\|^2, \end{aligned}$$

we obtain

$$\partial_t E(\tilde{u}_{M,N}, t) \leqslant d(u_{M,N}) E(\tilde{u}_{M,N}, t) + \left\|\tilde{f}(t)\right\|^2, \tag{4.15}$$

where $d(u_{M,N})$ is a positive constant depending only on $\|u_{M,N}\|_{L^\infty(0,T;L^\infty(\Omega))}$. Let

$$\begin{aligned} &\rho(\tilde{u}_{M,N,0}, \tilde{u}_{M,N,1}, \tilde{f}, t) \\ &\quad = \frac{1}{4}\|\tilde{u}_{M,N,0}\|^2_{1,\alpha,\beta,0,0} + \frac{1}{2}\|\tilde{u}_{M,N,0}\|^4_{L^4} + \|\tilde{u}_{M,N,1}\|^2 + \int_0^t \left\|\tilde{f}(s)\right\|^2 ds. \end{aligned}$$

Integrating (4.15) with respect to t, we obtain the following result.

Theorem 4.2. *Let $u_{M,N}$ be the solution of* (4.13)*, and $\tilde{u}_{M,N}$ be its error induced by $\tilde{u}_{M,N,0}$, $\tilde{u}_{M,N,1}$ and $\tilde{f}$. If $ab \leqslant \frac{3}{4}$, then for all $0 \leqslant t \leqslant T$,*

$$E(\tilde{u}_{M,N}, t) \leqslant \rho(\tilde{u}_{N,0}, \tilde{u}_{N,1}, \tilde{f}, t)e^{d(u_{M,N})t}.$$

We next deal with the convergence of the scheme (4.13). To obtain better error estimate, we compare the numerical solution $u_{M,N}$ with $U_{M,N} = \hat{P}^1_{M,N,\alpha,\beta,0,0}U$. By (4.12),

$$\begin{aligned}&\left(\partial_t^2 U_{M,N}(t) + U^3_{M,N}(t), \phi\right) + \hat{a}^{\nu}_{\alpha,\beta,0,0}\left(U_{M,N}(t), \phi\right) + \sum_{j=1}^{3}\left(G_j(t), \phi\right)\\ &\quad= \left(\nu_0 U_{M,N}(t) + f(t), \phi\right), \quad \forall \phi \in V_{M,N},\ t \in (0, T],\end{aligned} \tag{4.16}$$

with

$$\begin{aligned}&G_1(t, \phi) = \partial_t^2 U(t) - \partial_t^2 U_{M,N}(t), \qquad G_2(t, \phi) = \nu_0 U_{M,N}(t) - \nu_0 U(t),\\ &G_3(t, \phi) = U^3(t) - U^3_{M,N}(t).\end{aligned}$$

Furthermore, let $\tilde{U}_{M,N} = u_{M,N} - U_{M,N}$. Then subtracting (4.16) from (4.13) gives that

$$\begin{aligned}&\left(\partial_t^2 \tilde{U}_{M,N}(t) + \tilde{U}^3_{M,N}(t) + \tilde{G}_0(t), \phi\right) + \hat{a}^{\nu}_{\alpha,\beta,0,0}\left(\tilde{U}_{M,N}(t), \phi\right)\\ &\quad= \left(\nu_0 \tilde{U}_{M,N}(t), \phi\right) + \sum_{j=1}^{3}\left(G_j(t), \phi\right), \quad \forall \phi \in V_{M,N},\ t \in (0, T],\end{aligned} \tag{4.17}$$

with

$$\begin{aligned}&\tilde{G}_0(t) = 3\tilde{U}^2_{M,N}(t)U_{M,N}(t) + 3\tilde{U}_{M,N}(t)U^2_{M,N}(t),\\ &\partial_t \tilde{U}_{M,N}(0) = P_{M,N,0,0}U_1 - \hat{P}^1_{M,N,\alpha,\beta,0,0}U_1,\\ &\tilde{U}_{M,N}(0) = P_{M,N,0,0}U_0 - \hat{P}^1_{M,N,\alpha,\beta,0,0}U_0.\end{aligned}$$

Comparing (4.14) with (4.17), we build up an error estimation similar to (4.15). But $u_{M,N}$, $\tilde{u}_{M,N}$, $\tilde{u}_{M,N,0}$, $\tilde{u}_{M,N,1}$ and $\|u_{M,N}\|_{L^\infty(0,T;L^\infty(\Omega))}$ in (4.15) are now replaced by $U_{M,N}$, $\tilde{U}_{M,N}$, $\tilde{U}_{M,N}(0)$, $\partial_t \tilde{U}_{M,N}(0)$ and $\|U_{M,N}\|_{L^\infty(0,T;L^\infty(\Omega))}$, respectively. Thus it remains to estimate $\|G_j(t)\|$, $\|\partial_t \tilde{U}_{M,N}(0)\|$, $\|\tilde{U}_{M,N}(0)\|_{1,\alpha,\beta,0,0}$ and $\|\tilde{U}_{M,N}(0)\|_{L^4}$. Firstly, we have from Theorem 3.7 that

$$\begin{aligned}\left\|G_1(t)\right\| + \left\|G_2(t)\right\| &= \left\|\partial_t^2 U_{M,N}(t) - \partial_t^2 U(t)\right\|\\ &\leqslant c\left(M^{1-r} + N^{1-s}\right)\left(\left\|\partial_t^2 U(t)\right\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}} + \left\|U(t)\right\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}}\right).\end{aligned}$$

Next, by Remark 3.1, the imbedding theorem and Theorems 3.7 and 3.8, we get that for $d > 1$,

$$\begin{aligned}\left\|G_3(t)\right\| &\leqslant c\left(\left\|U_{M,N}(t)\right\|^2_\infty + \left\|U(t)\right\|^2_\infty\right)\left\|U_{M,N}(t) - U(t)\right\|\\ &\leqslant cM(U)\left(M^{1-r} + N^{1-s}\right)\left\|U(t)\right\|_{\hat{M}^{r,s}_{\alpha,\beta,0,0}},\end{aligned}$$

where

$$M(U) = \max_{0 \leqslant t \leqslant T}\left(\left\|U(t)\right\|^2_{L^2(\Lambda_2; H^{1+d}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1) \cap H^d(\Lambda_1))} + \left\|U(t)\right\|^2_{H^d(\Omega)}\right).$$

For simplicity, let integers $r,s \geqslant 1$ and $Z^{r,s}_{\alpha,\beta}(\Omega) = H^{r-1,s-1}_{\chi^{(0,0)},A}(\Omega) \cap \hat{M}^{r,s}_{\alpha,\beta,0,0}(\Omega)$. Its norm $\|v\|_{Z^{r,s}_{\alpha,\beta}}$ is defined in the usual way. Then by Theorems 3.3 and 3.7,

$$\begin{aligned}\|\partial_t \tilde{U}_{M,N}(0)\| &\leqslant \|P_{M,N,0,0,0,0}U_1 - U_1\| + \|\hat{P}^1_{M,N,\alpha,\beta,0,0}U_1 - U_1\| \\ &\leqslant c\big(M^{1-r} + N^{1-s}\big)\|U_1\|_{Z^{r,s}_{\alpha,\beta}}.\end{aligned}$$

Since $\alpha_1 = \beta_1 = 2$ and $\alpha_2 = \beta_2 = 0$, we use Theorems 3.2, 3.3 and 3.7 to derive that

$$\begin{aligned}\|\tilde{U}_{M,N}(0)\|^2_{1,\alpha,\beta,0,0} &\leqslant \|\tilde{U}_{M,N}(0)\|^2_{H^{1,1}_{\chi^{(\alpha,\beta)}}} + \|\tilde{U}_{M,N}(0)\|^2 \leqslant c(M+N)^2\|\tilde{U}_{M,N}(0)\|^2 \\ &\leqslant c(M+N)^2\big(M^{-r} + N^{-s}\big)^2\|U_0\|^2_{Z^{r+1,s+1}_{\alpha,\beta}}.\end{aligned}$$

Moreover, by Theorems 3.1, 3.3 and 3.7,

$$\|\tilde{U}_{M,N}(0)\|_{L^4} \leqslant c\big(M^2N\big)^{1/4}\|\tilde{U}_{M,N}(0)\| \leqslant c\big(M^2N\big)^{1/4}\big(M^{-r} + N^{-s}\big)\|U_0\|_{Z^{r+1,s+1}_{\alpha,\beta}}.$$

A combination of the previous estimates leads to the following result.

Theorem 4.3. *Let U and $u_{M,N}$ be the solutions of* (4.12) *and* (4.13), *respectively. Assume that for $\alpha_1 = \beta_1 = 2$, $\alpha_2 = \beta_2 = 0$, $d > 1$ and integers $r,s \geqslant 1$,*

$$\begin{aligned}&U \in L^\infty\Big(0,T; H^d_p(\Omega) \cap L^2_p\Big(\Lambda_2; H^{1+d}_{\chi^{(\frac{\alpha_1}{2},\frac{\beta_1}{2})},*}(\Lambda_1) \cap H^d(\Lambda_1)\Big)\Big) \cap H^2\big(0,T; \hat{M}^{r,s}_{p,\alpha,\beta,0,0}(\Omega)\big),\\ &U_0 \in Z^{r+1,s+1}_{p,\alpha,\beta}(\Omega), \qquad U_1 \in Z^{r,s}_{p,\alpha,\beta}(\Omega).\end{aligned}$$

Then for all $0 \leqslant t \leqslant T$,

$$\begin{aligned}E(u_{M,N} - U, t) \leqslant b^*\Big(&\big(M^{1-r} + N^{1-s}\big)^2 + (M+N)^2\big(M^{-r} + N^{-s}\big)^2 \\ &+ \big(M^2N\big)\big(M^{-r} + N^{-s}\big)^4\Big),\end{aligned}$$

b^* being a positive constant depending on the norms of U, U_0, U_1 and f in the mentioned spaces.

Remark 4.1. If $M = O(N)$, then the result in Theorem 4.3 is reduced to $E(u_{M,N} - U, t) \leqslant b^*(M^{1-r} + N^{1-s})^2$. On the other hand, if we take $u_{M,N}(0) = \hat{P}^1_{M,N,0,0,0,0}U_0$ and $\partial_t u_{M,N}(t) = \hat{P}^1_{M,N,0,0,0,0}U_1$, then we can weaken the conditions on U_0 and U_1 in Theorem 4.3, and obtain the same result.

The mixed Fourier–Jacobi approximation is also applicable to other problems. For instance, we consider the heat transfer inside a unit disc,

$$\begin{cases}\partial_t V - \frac{1}{r}\partial_r(r\partial_r V) - \frac{1}{r^2}\partial^2_\theta V = F, & (r,\theta,t) \in (0,1) \times [0,2\pi) \times (0,T],\\ V(1,\theta,t) = 0, & (\theta,t) \in [0,2\pi) \times [0,T],\\ V(r,\theta,0) = V_0(r,\theta), & (r,\theta) \in [0,1] \times [0,2\pi),\end{cases} \tag{4.18}$$

where all functions have the period 2π for the variable θ. In addition, the solution V satisfies the polar condition: $\partial_\theta V(0,\theta,t) = 0$. Let $r = \frac{1-x_1}{2}$, $\theta = x_2$, $x = (x_1,x_2)$, $\eta(x) = 1 - x_1$ and

$$U(x_1,x_2,t)=V\left(\frac{1-x_1}{2},x_2,t\right),\qquad U_0(x_1,x_2)=V_0\left(\frac{1-x_1}{2},x_2\right),$$
$$f(x_1,x_2,t)=F\left(\frac{1-x_1}{2},x_2,t\right).$$

Then problem (4.18) reads

$$\begin{cases}\eta\partial_t U-4\partial_{x_1}(\eta\partial_{x_1}U)-\frac{4}{\eta}\partial_{x_2}^2U=\eta f, & (x,t)\in\Omega\times(0,T],\\ U(-1,x_2,t)=\partial_{x_2}U(1,x_2,t)=0, & (x_2,t)\in\Lambda_2\times[0,T],\\ U(x,0)=U_0(x), & x\in\Omega.\end{cases}\tag{4.19}$$

The mixed Fourier–Jacobi approximation can also be used for some exterior problems. As an example, we consider the Laplace equation outside a unit disc,

$$\begin{cases}-\frac{1}{r}\partial_r(r\partial_r V)-\frac{1}{r^2}\partial_\theta^2V=F, & (r,\theta)\in(1,+\infty)\times[0,2\pi),\\ V(1,\theta)=\lim_{r\to\infty}r^a\partial_rV(r,\theta)=0, & a<2,\ \theta\in[0,2\pi).\end{cases}\tag{4.20}$$

Let

$$r=\frac{2}{1-x_1},\qquad \theta=x_2,\qquad U(x_1,x_2)=V\left(\frac{2}{1-x_1},x_2\right),$$
$$f(x_1,x_2)=\frac{4}{(1-x_1)^3}F\left(\frac{2}{1-x_1},x_2\right).$$

Then (4.20) is changed into

$$\begin{cases}-\partial_{x_1}(\eta\partial_{x_1}U)-\eta\partial_{x_2}^2U=f, & (x_1,x_2)\in\Omega,\\ U(-1,x_2)=\lim_{x_1\to1}\eta^{2-a}\partial_{x_1}U(x_1,x_2)=0, & x_2\in\Lambda_2.\end{cases}\tag{4.21}$$

Clearly we may use the Fourier–Jacobi spectral method to solve (4.19) and (4.21).

In the end of this paper, we present some numerical results. We first consider problem (4.1) with $a_1(x)=a_2(x)=1-x_1^2$ and $a_0(x)=1$, and take the test function $U(x)=\arcsin(x_1)\sin(\frac{1}{10}x_1+2x_2)$. Clearly, $|\partial_{x_1}U|\to\infty$ as $|x_1|\to1$. We use scheme (4.7) with $\alpha_i=\beta_i=1$, $i=1,2$, and $\gamma=\delta=0$, to solve (4.1) numerically. For description of the errors, let $\Lambda_{1,M}=\{x_1^j\mid 0\leqslant j\leqslant M\}$ be the set of Gauss–Legendre nodes, $\Lambda_{2,N}=\{x_2^j=\frac{\pi j}{N}\mid 0\leqslant j\leqslant 2N\}$, and

$$E_{\mathrm{abs}}(u_{M,N})=\max_{x\in\Lambda_{1,M}\times\Lambda_{2,N}}\left|U(x)-u_{M,N}(x)\right|,$$
$$E_{\mathrm{rel}}(u_{M,N})=\max_{x\in\Lambda_{1,M}\times\Lambda_{2,N}}\left|\frac{U(x)-u_{M,N}(x)}{U(x)}\right|.$$

We plot in Fig. 1 the errors with different M and N, which show the convergence of (4.7).

We next consider problem (4.8) and (4.9) with $a=\frac{1}{2}$, and take f in such a way that (4.12) has the exact solution $U(x,t)=(1-x)^\gamma(1+x)^\delta\exp(\frac{1}{10}\sin(x_1+2x_2)+\frac{1}{10}t)$. Its regularity depends on the values of γ and δ, essentially. We use scheme (4.13) with $\alpha_1=\beta_1=2$, $\alpha_2=\beta_2=0$, $\nu_0=1$, $\nu_1=\frac{1}{4}$ and $\nu_2=1$, to solve (4.12) numerically. In actual computation, we need to discretize (4.13) in time t. To do this, let τ be the mesh size in time, $\mathcal{R}_\tau=\{t=k\tau\mid k=1,2,\ldots,[\frac{T}{\tau}]\}$, and

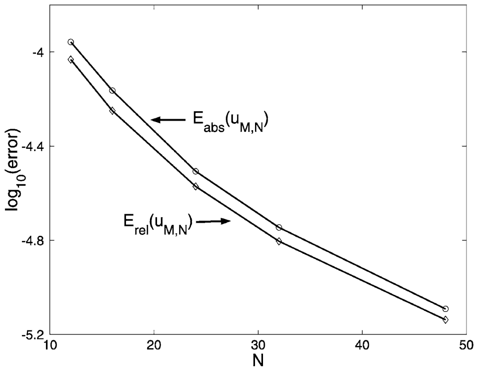

Fig. 1. The errors with different $M = N$.

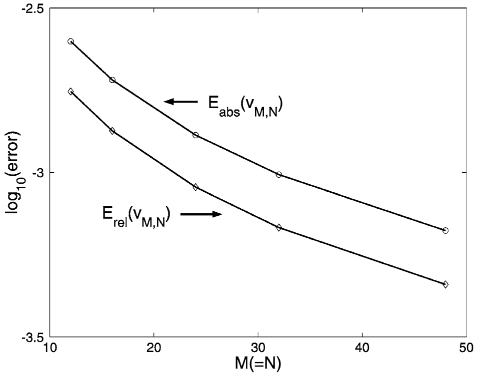

Fig. 2. The errors with various $M = N$.

$$\hat{v}(x,t) = \frac{1}{2}\big(v(x,t+\tau) + v(x,t-\tau)\big),$$
$$\hat{\mathcal{D}}_{\tau\tau} v(x,t) = \frac{1}{\tau^2}\big(v(x,t+\tau) - 2v(x,t) + v(x,t-\tau)\big).$$

To increase the stability of computation, we approximate the nonlinear term in (4.12) by

$$\hat{G}\big(v(x,t)\big) = \frac{1}{4}\sum_{j=0}^{3} v^j(x,t+\tau)v^{3-j}(x,t-\tau).$$

The fully discrete scheme is

$$\big(\hat{\mathcal{D}}_{\tau\tau} u_{M,N}(t) + \hat{G}\big(u_{M,N}(t)\big), \phi\big) + \hat{a}^{\nu}_{\alpha,\beta}\big(\hat{u}_{M,N}(t), \phi)\big) = \big(\nu_0 \hat{u}_{M,N}(t) + \hat{f}(t), \phi\big),$$
$$\forall \phi \in V_{M,N},\ t \in \mathcal{R}_\tau, \tag{4.22}$$

with

$$u_{M,N}(0) = P_{M,N,0,0}U_0, \qquad u_{M,N}(\tau) = P_{M,N,0,0}U_0 + \tau P_{M,N,0,0}U_1.$$

Now, let $y_1(x_1)$ and $y_2(x_2)$ be the same as in (4.10) with $b = 1$. The numerical solution of the original problem is $v_{M,N}(y_1, y_2, t) = u_{M,N}(\frac{e^{y_1}-1}{e^{y_1}+1}, y_2, t)$. Let

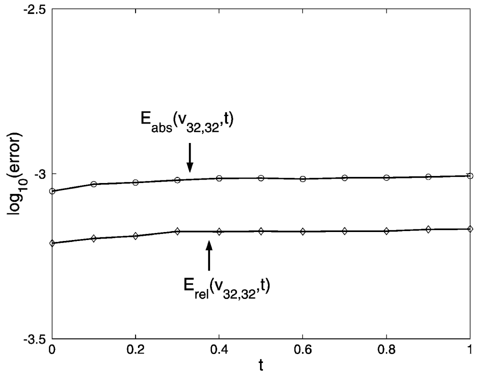

Fig. 3. The errors with various t.

$$\Lambda_{1,M}^{*}=\left\{y_1^j=y_1\left(x_1^j\right)\,\middle|\,x_1^j\in\Lambda_{1,M},\ 0\leqslant j\leqslant M\right\},$$
$$\Lambda_{2,N}^{*}=\left\{y_2^j=y_2\left(x_2^j\right)\,\middle|\,x_2^j\in\Lambda_{2,M},\ 0\leqslant j\leqslant 2N\right\}.$$

Denote by $E_{\mathrm{abs}}(v_{M,N},t)$ and $E_{\mathrm{rel}}(v_{M,N},t)$ the absolute error and the relative error of $V(y,t)-v_{M,N}(y,t)$ on the grid set $\Lambda_{1,M}^{*}\times\Lambda_{2,N}^{*}$ at time t. We plot in Fig. 2 the errors with $\gamma=\delta=10^{-2}$, $\tau=10^{-3}$ and $t=1$, which show the convergence of scheme (4.22). In Fig. 3, we plot the errors with $\gamma=\delta=10^{-2}$, $\tau=10^{-3}$ and $M=N=32$ in different time t, which demonstrate the stability of computation.

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