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Optimal Spectral-Galerkin Methods Using Generalized Jacobi Polynomials

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We extend the definition of the classical Jacobi polynomials withindexes $\alpha, \beta > -1$ to allow α and/or β to be negative integers. We show that the generalized Jacobi polynomials, with indexes corresponding to the number of boundary conditions in a given partial differential equation, are the natural basis functions for the spectral approximation of this partial differential equation. Moreover, the use of generalized Jacobi polynomials leads to much simplified analysis, more precise error estimates and well conditioned algorithms.

KEY WORDS: Generalized Jacobi polynomials; spectral-Galerkin method; high-order differential equations.

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1. GENERALIZED JACOBI POLYNOMIALS

The classical Jacobi polynomials, denoted by $J_n^{\alpha,\beta}(x)\,(n \geqslant 0,\ \alpha, \beta > -1)$ (cf. [18]), have been used extensively in mathematical analysis and practical applications, and play an important role in the analysis and implementation of spectral methods.

Recently, Shen [17] introduced an efficient spectral dual-Petrov-Galerkin method for third and higher odd-order differential equations, and pointed out that the basis functions used in [17], which are compact combinations of Legendre polynomials, can be viewed as generalized Jacobi polynomials with negative integer indexes, and their use not only simplified

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the numerical analysis for the spectral approximations of higher odd-order differential equations, but also led to very efficient numerical algorithms. In fact, the basis functions used in [16], which are compact combinations of Legendre polynomials, can also be viewed as generalized Jacobi polynomials with indexes $\alpha, \beta \leqslant -1$. Furthermore, the special cases with $(\alpha, \beta) = (-1, 0)$, $(-1, -1)$ have also been studied in [5, 10]. Hence, instead of developing approximation results for each particular pair of indexes, it would be very useful to carry out a systematic study on Jacobi polynomials with general negative integer indexes which can then be directly applied to other applications. It is with this motivation that we introduce in this paper a family of generalized Jacobi polynomials/functions with indexes α, $\beta \in \mathbb{R}$.

Let $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. We denote by $L^2_{\omega^{\alpha,\beta}}(I)$ $(I := (-1, 1))$ the weighted L^2 space with inner product:

$$(u, v)_{\omega^{\alpha,\beta}} := \int_I u(x)v(x)\omega^{\alpha,\beta}(x)dx, \tag{1.1}$$

and the associated norm $\|u\|_{\omega^{\alpha,\beta}} = (u, u)^{\frac{1}{2}}_{\omega^{\alpha,\beta}}$. Two of the most important properties of the classical Jacobi polynomials are: (i) they are mutually orthogonal in $L^2_{\omega^{\alpha,\beta}}(I)$, i.e.,

$$\int_I J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx = 0, \quad \forall n \neq m; \tag{1.2}$$

and (ii) $\{J_n^{\alpha,\beta}\}$ satisfy the recursive relation:

$$\partial_x J_n^{\alpha,\beta}(x) = C_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta+1}(x), \quad n \geqslant 1, \tag{1.3}$$

where $C_n^{\alpha,\beta} = \frac{1}{2}(n+\alpha+\beta+1)$. The restriction "$\alpha, \beta > -1$" was imposed so that the weight function $\omega^{\alpha,\beta} \in L^1(I)$.

We are interested in defining Jacobi polynomials with indexes α and/or $\beta \leqslant -1$, referred hereafter as generalized Jacobi polynomials (GJPs), in such a way that they satisfy some selected properties that are essentially relevant to spectral approximations. In this work, we shall restrict our attention to the cases when α and β are negative integers. The general cases are much more involved and will be investigated separately in [11].

Let $k, l \in \mathbb{Z}$ (the set of all integers), and define

$$J_n^{k,l}(x) = \begin{cases} (1-x)^{-k}(1+x)^{-l} J_{n-n_0}^{-k,-l}(x), & n_0 := -(k+l), & \text{if } k, l \leqslant -1, \\ (1-x)^{-k} J_{n-n_0}^{-k,l}(x), & n_0 := -k, & \text{if } k \leqslant -1,\ l > -1, \\ (1+x)^{-l} J_{n-n_0}^{k,-l}(x), & n_0 := -l, & \text{if } k > -1,\ l \leqslant -1, \\ J_{n-n_0}^{k,l}(x), & n_0 := 0, & \text{if } k, l > -1. \end{cases} \tag{1.4}$$

An important fact is that the so-defined GJPs satisfy (1.2) and (1.3). It can also be easily verified that $\{J_n^{k,l}: n \geqslant n_0\}$ forms a complete orthogonal system in $L^2_{\omega^{k,l}}(I)$. Hence, we define

$$Q_N^{k,l} := \mathrm{span}\{J_{n_0}^{k,l}, J_{n_0+1}^{k,l}, \ldots, J_N^{k,l}\}, \quad k, l \in \mathbb{Z}, \tag{1.5}$$

and consider the orthogonal projection $\pi_N^{k,l}: L^2_{\omega^{k,l}}(I) \to Q_N^{k,l}$ defined by

$$(u - \pi_N^{k,l} u, v_N)_{\omega^{k,l}} = 0, \quad \forall v_N \in Q_N^{k,l}. \tag{1.6}$$

For real numbers α, β and $r \in \mathbb{N}$ (the set of all nonnegative integers), we define the space

$$H^r_{\omega^{\alpha,\beta}, A}(I) := \{u : u \text{ is measurable on } I \text{ and } \|u\|_{r,\omega^{\alpha,\beta},A} < \infty\}, \tag{1.7}$$

equipped with the norm and semi-norm

$$\|u\|_{r,\omega^{\alpha,\beta},A} = \Big(\sum_{k=0}^{r} \|\partial_x^k u\|^2_{\omega^{\alpha+k,\beta+k}} \Big)^{\frac{1}{2}}, \quad |u|_{r,\omega^{\alpha,\beta},A} = \|\partial_x^r u\|_{\omega^{\alpha+r,\beta+r}},$$

where $\|v\|^2_\omega = \int_I v^2 \omega dx$. Hereafter, we denote by c a generic positive constant independent of any function and N, and use the expression $A \lesssim B$ to mean that there exists a generic positive constant c such that $A \leqslant cB$.

The following theorem is a direct extension of the same result for $k, l > -1$ (see, for instance [8, 10]; see also [5] for the special case $k = 0$ and $l = -1$) and can be proved in a similar fashion thanks to (1.2) and (1.3).

Theorem 1.1. Let $k, l \in \mathbb{Z}$. Then for any $u \in H^r_{\omega^{k,l},A}(I)$, $r \in \mathbb{Z}$, $r \geqslant 1$ and $0 \leqslant \mu \leqslant r$,

$$\|\pi_N^{k,l} u - u\|_{\mu,\omega^{k,l},A} \lesssim N^{\mu - r} |u|_{r,\omega^{k,l},A}. \tag{1.8}$$

An important property of the GJPs is that for $k, l \in \mathbb{Z}$ and $k, l \geqslant 1$,

$$\begin{aligned} &\partial_x^i J_n^{-k,-l}(1) = 0, \ \ i = 0, 1, \ldots, k-1; \\ &\partial_x^j J_n^{-k,-l}(-1) = 0, \ \ j = 0, 1, \ldots, l-1. \end{aligned} \tag{1.9}$$

Hence, $\{J_n^{-k,-l}\}$ are natural candidates as basis functions for PDEs with the following boundary conditions:

$$\begin{aligned} &\partial_x^i u(1) = a_i, \quad i = 0, 1, \ldots, k-1; \\ &\partial_x^j u(-1) = b_j, \quad j = 0, 1, \ldots, l-1. \end{aligned} \tag{1.10}$$

For example, we can easily verify that

$$J_n^{-1,-1}(x) = \gamma_n^{-1,-1}(L_{n-2}(x) - L_n(x)), \quad n \geqslant 2, \tag{1.11}$$

$$J_n^{-2,-2}(x) = \gamma_n^{-2,-2}\left(L_{n-4}(x) - \frac{2(2n-3)}{2n-1}L_{n-2}(x) + \frac{2n-5}{2n-1}L_n(x)\right), \quad n \geqslant 4, \tag{1.12}$$

where $L_k(x)$ is the Legendre polynomial of kth degree and $\gamma_n^{\alpha,\beta}$ are normalization constants to be specified. We note that the right-hand sides of (1.11) and (1.12) were exactly the basis functions used in [16] for the second- and forth-order equations. However, the fact that they can be identified as the GJPs $J_n^{-1,-1}$ and $J_n^{-2,-2}$ was not recognized there. Since the GJPs satisfy all given boundary conditions of the underlying problem,there is no need to construct special quadratures (which become increasingly complicated as the order of the differential equations increase) involving derivatives at end-points as in [12, 13] for third-order equations and in [3] for fourth-order equations. Unlike in a collocation method, the implementation for higher-order differential equations is cumbersome and very ill-conditioned, and the analysis is very difficult and may not lead to optimal error estimates (see [3] for an example on a fourth-order equation), as we shall demonstrate below, the spectral approximations using GJPs lead to well-conditioned system, sparse for problems with constant coefficients (cf. [16, 17]), that can be efficiently implemented. Furthermore, using the GJPs greatly simplifies the analysis and leads to more precise error estimates.

2. SPECTRAL-GALERKIN METHODS FOR EVEN-ORDER EQUATIONS

We consider the following $2m$th order linear equation:

$$\begin{aligned} &u^{(2m)}(x) + \sum_{k=0}^{2m-1} b_{2m-k}(x)u^{(k)}(x) = f(x), \quad \text{in } I, \\ &u^{(k)}(\pm 1) = 0, \quad 0 \leqslant k \leqslant m-1. \end{aligned} \tag{2.1}$$

We introduce the bilinear form associated with (2.1):

$$\begin{aligned} a_m(u,v) = \ & (\partial_x^m u, \partial_x^m v) + (-1)^m(\partial_x^{m-1}u, \partial_x^m(b_1 v)) \\ & + (-1)^{m-1}(\partial_x^{m-1}u, \partial_x^{m-1}(b_2 v)) + \cdots + (b_{2m}u, v), \quad \forall u, v \in H^m(I). \end{aligned} \tag{2.2}$$

Let $\|\cdot\|_m$ and $|\cdot|_m$ denote the norm and semi-norm in $H^m(I)$ and $H_0^m(I)$, respectively. As usual, we assume that the bilinear form is continuous and elliptic in $H_0^m(I)$, i.e.,

$$|a_m(u,v)| \leqslant C_0|u|_m|v|_m, \quad \forall u,v\in H_0^m(I), \tag{2.3a}$$

$$a_m(u,u) \geqslant C_1|u|_m^2, \quad \forall u\in H_0^m(I). \tag{2.3b}$$

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N and $V_N := \mathcal{P}_N \cap H_0^m(I)$. The spectral-Galerkin approximation to (2.1) is: Find $u_N \in V_N$ such that

$$a_m(u_N, v_N) = (f, v_N), \qquad \forall v_N \in V_N. \tag{2.4}$$

Let us denote $\pi_N^m = \pi_N^{-m,-m}$. We note immediately that

$$\begin{aligned}(\partial_x^m(\pi_N^m u - u), \partial_x^m v_N) &= (-1)^m(\pi_N^m u - u, \partial_x^{2m} v_N)\\ &= (-1)^m(\pi_N^m u - u, \omega^{m,m}\partial_x^{2m} v_N)_{\omega^{-m,-m}} = 0, \quad \forall v_N \in V_N,\end{aligned} \tag{2.5}$$

which is a consequence of (1.6) and the fact that $\omega^{m,m}\partial_x^{2m} v_N \in V_N$. In other words, π_N^m is simultaneously the orthogonal projector associated with $(\cdot,\cdot)_{\omega^{-m,-m}}$ and $(\partial_x^m\cdot, \partial_x^m\cdot)$. Thanks to the above property and Theorem 1.1, one can prove using a rather standard procedure the following result (cf. [11] for details):

Theorem 2.1. If $u\in H_0^m(I)\cap H^r_{\omega^{-m,-m},A}(I)$, $m,\mu,r\in\mathbb{Z}$, $1\leqslant m\leqslant r$ and $0\leqslant\mu\leqslant m$, then

$$\|u-u_N\|_\mu \lesssim N^{\mu-r}|u|_{r,\omega^{-m,-m},A}. \tag{2.6}$$

Remark 2.1. The above result is more precise than the best error estimate

$$\|u-u_N\|_m \lesssim N^{m-r}\|u\|_r, \quad 0\leqslant m\leqslant r, \tag{2.7}$$

which could have obtained by using the H_0^m−orthogonal projection results in [4]. Since $|u|_{r,\omega^{-m,-m},A} = \|\partial_x^r u\|_{\omega^{r-m,r-m}}$ and $r \geqslant m$, the result (2.6) is sharper than (2.7). In particular, Theorem 2.1 can be used to derive improved error estimates for solutions with singularities at the end points. As an example, let

$$u(x) = (1-x)^\gamma v(x), \quad v\in C^\infty(I), \quad \gamma > \frac{1}{2}, \qquad x\in I, \tag{2.8}$$

be a solution of (2.1) with $m=1$. On can check that for any $\varepsilon>0$, $u\in H^{2\gamma+1-\varepsilon}_{\omega^{0,0},A}(I)$, but when measured in the standard Sobolev norm, we only have $u\in H^{\gamma+\frac{1}{2}-\varepsilon}(I)$. Hence, Theorem 2.1 with $m=1$ implies that

$$\|u-u_N\|_1 \lesssim N^{\varepsilon-2\gamma}, \tag{2.9}$$

while the usual analysis (cf. [3]) only leads to

$$\|u-u_N\|_1 \lesssim N^{\varepsilon-\gamma+\frac{1}{2}}. \tag{2.10}$$

Although the estimate (2.9) has been established before by other means (see, for instance, [2, 7, 1, 6] among others), the approach using GJPs is straightforward and can be directly carried over to higher-order equations.

Matrix form of (2.4):

In view of the homogeneous boundary conditions satisfied by $J_k^{-m,-m}$, we have

$$V_N=\operatorname{span}\{J_{2m}^{-m,-m}, J_{2m+1}^{-m,-m},\dots,J_N^{-m,-m}\}.$$

Using the facts that $\omega^{m,m}\partial_x^{2m}J_l^{-m,-m}\in V_l$ and $J_k^{-m,-m}$ is orthogonal to V_l if $k>l$, we find that

$$\begin{aligned}(\partial_x^m J_k^{-m,-m},\partial_x^m J_l^{-m,-m}) &=(-1)^m(J_k^{-m,-m},\partial_x^{2m}J_l^{-m,-m})\\ &=(J_k^{-m,-m},\omega^{m,m}\partial_x^{2m}J_l^{-m,-m})_{\omega^{-m,-m}}=0.\end{aligned} \tag{2.11}$$

By symmetry, the same is true if $k<l$. Hence, we can properly scale $\phi_k(x):=J_k^{-m,-m}$ such that

$$(\partial_x^m\phi_k,\partial_x^m\phi_l)=\delta_{kl}.$$

Hence, by setting

$$f_k=(f,\phi_k),\quad u_N=\sum_{l=2m}^{N}\hat{u}_l\phi_l,\quad a_{kl}=a_m(\phi_l,\phi_k),\quad A=(a_{kl})_{2m\leqslant k,l\leqslant N},$$
$$\mathbf{f}=(f_{2m},f_{2m+1},\dots,f_N)^T,\quad \mathbf{u}=(\hat{u}_{2m},\hat{u}_{2m+1},\dots,\hat{u}_N)^T,$$

the matrix system associated with (2.4) becomes $A\mathbf{u}=\mathbf{f}$. Thanks to (2.3a)–(2.3b), we have

$$C_0\|\mathbf{u}\|_{l^2}^2=C_0|u_N|_m^2\leqslant a_m(u_N,u_N)=(A\mathbf{u},\mathbf{u})_{l^2}\leqslant C_1|u_N|_m^2=C_1\|\mathbf{u}\|_{l^2}^2, \tag{2.12}$$

which implies that $\mathrm{cond}(A) \leqslant C_1/C_0$ and is independent of N. It can be easily shown that A is a sparse matrix with bandwidth $2m+1$ if $\{b_j(x)\}$ are constants.

3. SPECTRAL DUAL-PETROV-GALERKIN METHODS FOR ODD-ORDER EQUATIONS

We consider the following $(2m+1)$th-order linear equation:

$$\begin{aligned} &(-1)^{m+1}u^{(2m+1)} + S_m(u) + \gamma u = f, \quad \text{in } I,\\ &u^{(k)}(\pm 1) = u^{(m)}(1) = 0, \quad 0 \leqslant k \leqslant m-1, \end{aligned} \tag{3.1}$$

where $S_m(u)$ is a linear combination of $u^{(j)}$, $1 \leqslant j \leqslant 2m-1$. Note that a semi-implicit time discretization of the KdV-type equations will lead to (3.1).

Because the leading differential operator in (3.1) is not symmetric, it is natural to use a dual-Petrov-Galerkin method, in which the trial functions satisfy the underlying boundary conditions of the differential equations, and the test functions fulfill the corresponding "dual" boundary conditions [17]. More precisely, for a given $m \geqslant 1$, we define

$$V = \left\{ u \in H^{m+1}(I) : u^{(k)}(\pm 1) = 0,\ 0 \leqslant k \leqslant m-1,\ u^{(m)}(1) = 0 \right\},$$

and the "dual" space:

$$V^* = \left\{ u \in H^{m+1}(I) : u^{(k)}(\pm 1) = 0,\ 0 \leqslant k \leqslant m-1,\ u^{(m)}(-1) = 0 \right\}.$$

Setting $V_N = V \cap \mathcal{P}_N$ and $V_N^* = V^* \cap \mathcal{P}_N$, the dual-Petrov-Galerkin approximation to (3.1) is: Find $u_N \in V_N$ such that

$$-(\partial_x^{m+1}u_N, \partial_x^m v_N) + (S_m(u_N), v_N) + \gamma(u_N, v_N) = (f, v_N), \quad \forall v_N \in V_N^*. \tag{3.2}$$

Thanks to the homogeneous boundary conditions built in V_N and V_N^*, we observe from the definitions of GJPs that

$$\begin{aligned} V_N &= Q_N^{-m-1,-m} := \mathrm{span}\{J_{2m+1}^{-m-1,-m}, J_{2m+2}^{-m-1,-m}, \ldots, J_N^{-m-1,-m}\},\\ V_N^* &= Q_N^{-m,-m-1} := \mathrm{span}\{J_{2m+1}^{-m,-m-1}, J_{2m+2}^{-m,-m-1}, \ldots, J_N^{-m,-m-1}\}. \end{aligned} \tag{3.3}$$

For example, one can verify that

$$\begin{aligned} J_n^{-2,-1}(x) = \gamma_n^{-2,-1}\Big(&L_{n-3}(x) - \frac{2n-3}{2n-1}L_{n-2}(x)\\ &- L_{n-1}(x) + \frac{2n-3}{2n-1}L_n(x)\Big), \quad n \geqslant 3, \end{aligned} \tag{3.4}$$

$$J_n^{-1,-2}(x) = \gamma_n^{-1,-2}\Big(L_{n-3}(x) + \frac{2n-3}{2n-1}L_{n-2}(x) - L_{n-1}(x) - \frac{2n-3}{2n-1}L_n(x)\Big), \quad n \geqslant 3. \tag{3.5}$$

Note that for any $v_N \in V_N$, we have $\omega^{-1,1}v_N \in V_N^*$. Hence, we can rewrite the formulation (3.2) into the following weighted spectral-Galerkin form: Find $u_N \in V_N$ such that

$$\begin{aligned} b_m(u_N, u_N) :&= -(\partial_x^{m+1}u_N, \omega^{1,-1}\partial_x^m(\omega^{-1,1}v_N))_{\omega^{-1,1}} + (S_m(u_N), v_N)_{\omega^{-1,1}} \\ &\quad + \gamma(u_N, v_N)_{\omega^{-1,1}} = (f, v_N)_{\omega^{-1,1}}, \quad \forall v_N \in V_N. \end{aligned} \tag{3.6}$$

As demonstrated below, the dual-Petrov-Galerkin formulation (3.2) is preferable in numerical implementation, while the weighted Galerkin scheme (3.6) is more convenient for numerical analysis.

The following "coercivity" property ensures that (3.2) is well-posed.

Lemma 3.1.

$$\begin{aligned} &-(\partial_x^{m+1}u, \partial_x^m(u\omega^{-1,1})) \\ &\quad = (2m+1)\int_I \Big(\partial_x^m\Big(\frac{u}{1-x}\Big)\Big)^2 dx, \quad \forall u \in V_N = Q_N^{-m-1,-m}. \end{aligned} \tag{3.7}$$

Proof. For any $u \in V_N$, we set $u = (1-x)\Phi$ with $\Phi \in Q_{N-1}^{-m,-m}$. Then integrating by parts yields that

$$\begin{aligned} &-(\partial_x^{m+1}u, \partial_x^m(u\omega^{-1,1})) \\ &\quad = -((1-x)\partial_x^{m+1}\Phi - (m+1)\partial_x^m\Phi, (1+x)\partial_x^m\Phi + m\partial_x^{m-1}\Phi) \\ &\quad = -\frac{1}{2}\int_I \partial_x\{(\partial_x^m\Phi)^2\}(1-x^2)dx + (m+1)\int_I (\partial_x^m\Phi)^2(1+x)dx \\ &\qquad + \frac{m(m+1)}{2}\int_I \partial_x\{(\partial_x^{m-1}\Phi)^2\}dx + m\int_I \partial_x^m\Phi\,\partial_x((1-x)\partial_x^{m-1}\Phi)dx \\ &\quad = -\int_I (\partial_x^m\Phi)^2 x\,dx + (m+1)\int_I (\partial_x^m\Phi)^2(1+x)dx \\ &\qquad + m\int_I (\partial_x^m\Phi)^2(1-x)dx - \frac{m}{2}\int_I \partial_x\{(\partial_x^{m-1}\Phi)^2\}dx \\ &\quad = (2m+1)\int_I (\partial_x^m\Phi)^2 dx = (2m+1)\int_I \Big(\partial_x^m\Big(\frac{u}{1-x}\Big)\Big)^2 dx. \end{aligned}$$

$\square$

3.1. Error Estimates

Lemma 3.2. Let $\widetilde{\pi}_N^m := \pi_N^{-m-1,-m}$ be the orthogonal projection defined in (1.6). Then

$$(\partial_x^{m+1}(\widetilde{\pi}_N^m u - u), \partial_x^m v_N) = 0, \quad \forall u \in V, \quad v_N \in V_N^*, \tag{3.8}$$

$$(\partial_x^{m+1}(\widetilde{\pi}_N^m u - u), \omega^{1,-1}\partial_x^m(\omega^{-1,1} v_N))_{\omega^{-1,1}} = 0, \quad \forall u \in V,\ v_N \in V_N. \tag{3.9}$$

Proof. By the definition (1.6), for any $v_N \in V_N^*$,

$$(\partial_x^{m+1}(\widetilde{\pi}_N^m u - u), \partial_x^m v_N) = (-1)^{m+1}(\widetilde{\pi}_N^m u - u, \omega^{m+1,m}\partial_x^{2m+1} v_N)_{\omega^{-m-1,-m}} = 0.$$

Here, we used the fact that for any $v_N \in V_N^*$, $\omega^{m+1,m}\partial_x^{2m+1} v_N \in V_N$. Since for any $v_N \in V_N$, we have $\omega^{-1,1} v_N \in V_N^*$. Hence, (3.9) is a direct consequence of (3.8). □

The above lemma indicates that $\pi_N^{-m-1,-m}$ is simultaneously orthogonal projectors with respect to two bilinear forms.

To simplify the presentation, we only consider the case

$$S_m(u) = (-1)^m \delta\, u^{(2m-1)}, \tag{3.10}$$

(where δ is a non-negative constant) since other linear terms with derivatives lower than $2m-1$ can be treated similarly. In this case, we can conclude from Lemma 3.1 and the Lax-Milgram Lemma that (3.6) admits a unique solution when $\gamma > 0$ and $\delta \geqslant 0$. Furthermore, we can derive from Poincaré inequality that there exists $C_2 > 0$ such that

$$\begin{aligned}(2m+1)\int_I \left(\partial_x^m\left(\frac{u_N}{1-x}\right)\right)^2 dx &\leqslant b_m(u_N, u_N) \\ &\leqslant C_2(2m+1)\int_I \left(\partial_x^m\left(\frac{u_N}{1-x}\right)\right)^2 dx, \quad \forall u_N \in V_N.\end{aligned} \tag{3.11}$$

Let us denote $\hat{e}_N = \widetilde{\pi}_N^m u - u_N$ and $e_N = u - u_N = (u - \widetilde{\pi}_N^m u) + \hat{e}_N$.

Lemma 3.3. Let $\gamma, \delta > 0$. For $u \in V \cap H^r_{\omega^{-m-1,-m},A}(I)$ with integer $r \geqslant m+1$, we have

$$\begin{aligned}&\|\partial_x^m((1-x)^{-1}\hat{e}_N)\|^2 + \|\partial_x^{m-1}((1-x)^{-1}\hat{e}_N)\|^2 \\ &\quad + \|\hat{e}_N\|^2_{\omega^{-1,1}} \lesssim N^{2(m-r-1)}|u|^2_{r,\omega^{-m-1,-m},A}.\end{aligned} \tag{3.12}$$

Proof. Since for any $v_N \in V_N, \omega^{-1,1} v_N \in V_N^* \subset V^*$, we have from (3.1)–(3.10) and (3.6) that

$$-(\partial_x^{m+1} e_N, \omega^{1,-1}\partial_x^m(\omega^{-1,1}v_N))_{\omega^{-1,1}} - \delta(\partial_x^m e_N, \omega^{1,-1}\partial_x^{m-1}(\omega^{-1,1}v_N))_{\omega^{-1,1}} \\ +\gamma(e_N, v_N)_{\omega^{-1,1}} = 0, \quad \forall v_N \in V_N. \tag{3.13}$$

Taking $v_N = \hat{e}_N$ in the above relation, we obtain from (3.9) that

$$-(\partial_x^{m+1} \hat{e}_N, \omega^{1,-1}\partial_x^m(\omega^{-1,1}\hat{e}_N))_{\omega^{-1,1}} - \delta(\partial_x^m \hat{e}_N, \omega^{1,-1}\partial_x^{m-1}(\omega^{-1,1}\hat{e}_N))_{\omega^{-1,1}} \\ +\gamma\|\hat{e}_N\|^2_{\omega^{-1,1}} = -\delta(\partial_x^m(\widetilde{\pi}_N^m u - u), \omega^{1,-1}\partial_x^{m-1}(\omega^{-1,1}\hat{e}_N))_{\omega^{-1,1}} \\ +\gamma(\widetilde{\pi}_N^m u - u, \hat{e}_N)_{\omega^{-1,1}}. \tag{3.14}$$

We get from (1.8) that

$$\begin{aligned} |(\widetilde{\pi}_N^m u - u, \hat{e}_N)_{\omega^{-1,1}}| &\leqslant \|\widetilde{\pi}_N^m u - u\|_{\omega^{-1,1}}\|\hat{e}_N\|_{\omega^{-1,1}} \\ &\lesssim \|\widetilde{\pi}_N^m u - u\|_{\omega^{-m-1,-m}}\|\hat{e}_N\|_{\omega^{-1,1}} \\ &\leqslant \varepsilon\|\hat{e}_N\|^2_{\omega^{-1,1}} + \frac{c}{4\varepsilon}N^{-2r}|u|^2_{r,\omega^{-m-1,-m},A}. \end{aligned} \tag{3.15}$$

Setting $\hat{e}_N = (1-x)\hat{\phi}$ with $\hat{\phi} \in Q_{N-1}^{-m,-m}$, we find by integration by parts that

$$\begin{aligned} \int_I (\partial_x^m(\omega^{-1,1}\hat{e}_N))^2 dx &= \int_I (\partial_x^m((1+x)\hat{\phi}))^2 dx \\ &\leqslant 2\int_I (\partial_x^m\hat{\phi})^2(1+x)^2 dx + 2m^2\int_I (\partial_x^{m-1}\hat{\phi})^2 dx \\ &\leqslant 8\|\partial_x^m\hat{\phi}\|^2 + 2m^2\|\partial_x^{m-1}\hat{\phi}\|^2 \\ &= 8\|\partial_x^m((1-x)^{-1}\hat{e}_N)\|^2 + 2m^2\|\partial_x^{m-1}((1-x)^{-1}\hat{e}_N)\|^2. \end{aligned} \tag{3.16}$$

This fact with (1.8) leads to (for $\varepsilon > 0$)

$$\begin{aligned} &|(\partial_x^m(\widetilde{\pi}_N^m u - u), \partial_x^{m-1}(\omega^{-1,1}\hat{e}_N))| = |(\partial_x^{m-1}(\widetilde{\pi}_N^m u - u), \partial_x^m(\omega^{-1,1}\hat{e}_N))| \\ &\quad \leqslant \|\partial_x^{m-1}(\widetilde{\pi}_N^m u - u)\|_{\omega^{-2,-1}}\|\partial_x^m(\omega^{-1,1}\hat{e}_N)\|_{\omega^{2,1}} \\ &\quad \leqslant \|\partial_x^{m-1}(\widetilde{\pi}_N^m u - u)\|_{\omega^{-m-1,-m}}\|\partial_x^m(\omega^{-1,1}\hat{e}_N)\| \\ &\quad \leqslant \frac{4\varepsilon}{m^2}\|\partial_x^m((1-x)^{-1}\hat{e}_N)\|^2 + \varepsilon\|\partial_x^{m-1}((1-x)^{-1}\hat{e}_N)\|^2 \\ &\qquad + \frac{cm^2}{\varepsilon}N^{2(m-r-1)}|u|^2_{r,\omega^{-m-1,-m},A}. \end{aligned} \tag{3.17}$$

Finally, applying (3.7) to the left-hand side of (3.14), we obtain the desired result from the above estimates with a suitably small ε. □

Theorem 3.1. Let u and u_N be the solutions of (3.1)–(3.10) and (3.6), respectively. If $u \in V \cap H^r_{\omega^{-m-1,-m},A}(I)$, then for $\gamma, \delta > 0$, $m, r \in \mathbb{N}$, $m \geqslant 1$ and $r \geqslant m+1$,

$$\begin{aligned}&\|\partial_x^m((1-x)^{-1}(u-u_N))\|_{\omega^{1,0}} + N\|\partial_x^{m-1}((1-x)^{-1}(u-u_N))\| \\ &\quad + N\|u-u_N\|_{\omega^{-1,1}} \lesssim N^{m-r}|u|_{r,\omega^{-m-1,-m},A}.\end{aligned} \tag{3.18}$$

Proof. For any $v \in V$, a direct calculation yields

$$\partial_x^m((1-x)^{-1}v(x)) = \sum_{j=0}^{m} \frac{m!}{j!}(1-x)^{j-m-1}\partial_x^j v(x). \tag{3.19}$$

By the Hardy's inequality (see [9]), we derive that for $d < 1$,

$$\int_I v^2(x)(1-x)^{d-2}dx \lesssim \int_I (\partial_x v(x))^2(1-x)^d dx, \tag{3.20}$$

provided that $v(1) = 0$ and the right side of the inequality is finite. Thanks to the homogeneous boundary conditions built in V, we have from the above inequality that

$$\begin{aligned}&\int_I (\partial_x^j v(x))^2(1-x)^{2j-2m-1}dx \lesssim \int_I (\partial_x^{j+1}v(x))^2(1-x)^{2j-2m+1}dx \lesssim \cdots \\ &\quad \lesssim \int_I (\partial_x^m v(x))^2(1-x)^{-1}dx, \quad 0 \leqslant j \leqslant m-1.\end{aligned} \tag{3.21}$$

Therefore, by (1.8), (3.19) and (3.21),

$$\begin{aligned}&\|\partial_x^m((1-x)^{-1}(\widetilde{\pi}_N^m u - u))\|^2_{\omega^{1,0}} \lesssim \sum_{j=0}^{m}\int_I (\partial_x^j(\widetilde{\pi}_N^m u - u))^2(1-x)^{2j-2m-1}dx \\ &\lesssim \|\partial_x^m(\widetilde{\pi}_N^m u - u)\|^2_{\omega^{-1,0}} \lesssim N^{2(m-r)}|u|^2_{r,\omega^{-m-1,-m},A}.\end{aligned} \tag{3.22}$$

Similarly, we have

$$\begin{aligned}&\|\partial_x^{m-1}((1-x)^{-1}(\widetilde{\pi}_N^m u - u))\|^2 \lesssim \sum_{j=0}^{m-1}\int_I (\partial_x^j(\widetilde{\pi}_N^m u - u))^2(1-x)^{2j-2m}dx \\ &\quad \lesssim \|\partial_x^{m-1}(\widetilde{\pi}_N^m u - u)\|^2_{\omega^{-2,0}} \lesssim \|\partial_x^{m-1}(\widetilde{\pi}_N^m u - u)\|^2_{\omega^{-2,-1}} \\ &\quad \lesssim N^{2(m-r-1)}|u|^2_{r,\omega^{-m-1,-m},A}.\end{aligned} \tag{3.23}$$

Moreover, by (1.8),

$$\|\widetilde{\pi}_N^m u - u\|_{\omega^{-1,1}} \lesssim \|\widetilde{\pi}_N^m u - u\|_{\omega^{-m-1,-m}} \lesssim N^{-r}|u|_{r,\omega^{-m-1,-m},A}. \tag{3.24}$$

Hence, a combination of (3.12) and (3.22)–(3.24) leads to the desired result. □

Remark 3.1. For the sake of simplicity, the first two terms of our estimates in (3.18) were expressed in terms of $\partial_x^j((1-x)^{-1}(u-u_N))$. It is possible to obtain optimal estimates in terms of $\partial_x^j(u-u_N)$ by using Hardy-type inequalities (see Theorems 2.2 and 3.2 in [17] for examples with $m=1$ and $m=2$, respectively). However, this process could be tedious for $m>2$.

Remark 3.2. Using an argument similar to that in the even case, one can show that the linear system associated with (3.2) is well-conditioned (see [17] for a proof in the case $m=1$).

4. NUMERICAL RESULTS

We present below some numerical results to illustrate the efficiency of the proposed spectral methods using generalized Jacobi polynomials as basis functions. We shall only consider the odd-order equations since the implementations of even-order equations are similar and simpler. We first describe how to deal with variable coefficients, and make a comparison with the collocation method based on special quadratures involving derivatives at end-points (see [3, 12, 13] for third- and/or fourth-order equations). Then we apply our method to a fifth-order KdV-type equation.

4.1. Problems with Variable Coefficients

As an example, we consider the following problem:

$$\begin{aligned}&(-1)^{m+1}u^{(2m+1)}+F_m(u)=f, \quad \text{in } I,\\ &u^{(k)}(\pm 1)=u^{(m)}(1)=0, \quad 0\leqslant k\leqslant m-1,\end{aligned} \tag{4.1}$$

where

$$F_m(u)=\sum_{k=0}^{2m}a_k(x)u^{(k)}(x) \tag{4.2}$$

and $\{a_k\}_{k=0}^{2m}$ are given functions on $\bar{I}$.

To solve (4.1)–(4.2) numerically, we apply the *dual-Petrov-Galerkin method with numerical integrations*. For this purpose, let

$$(u,v)_N=\sum_{j=0}^{N}u(x_j)v(x_j)\omega_j, \quad \forall u,v\in C(\bar{I}),$$

be the discrete inner product associated with the Legendre–Gauss–Lobatto quadrature. We recall that

$$(u, v)_N = (u, v), \quad \forall uv \in \mathcal{P}_{2N-1}. \tag{4.3}$$

The dual-Petrov-Galerkin method with numerical integrations to (4.1)–(4.2) is to find $u_N \in V_N$ such that

$$-(\partial_x^{m+1} u_N, \partial_x^m v_N)_N + (F_m(u_N), v_N)_N = (f, v_N)_N, \quad \forall v_N \in V_N^*. \tag{4.4}$$

Here, the spaces V_N and V_N^* are defined in (3.3) with $m = 2$. Hence, using the same notations as in Section (3) and setting

$$p_{kl} = (F_m(\Phi_l), \Psi_k)_N, \quad P = (p_{kl})_{2m+1 \leqslant k,l \leqslant N},$$

the linear system (4.4) becomes

$$(I + P)\mathbf{u} = \mathbf{f}. \tag{4.5}$$

It can be shown, as in Section 3, that the above linear system is well conditioned under some mild conditions on F_m. Since the action of P upon a vector $\mathbf{u}$ can be evaluated in $O(N^2)$ operations without explicit knowledge of P, one can solve (4.5) in $O(N^2)$ operations by using a conjugate gradient type iterative method.

As a comparison, we compare the dual-Petrov-Galerkin method with a properly formulated collocation method which involves full matrix with large condition numbers (of order N^{2k} for kth order problem) and suffers from significant roundoff errors when N and especially k are large, as to be shown below. We note that although one can build effective preconditioners using finite difference or finite element approximations for the collocation matrices of order two or four (cf. [14]), there is no effective preconditioner for collocation matrices of odd-order.

As an example, we consider the following fifth-order equation:

$$u^{(5)} + a_1(x)u' + a_0(x)u = f, \quad \text{in } I, \quad u(\pm 1) = u'(\pm 1) = u''(1) = 0. \tag{4.6}$$

Note that the collocation method is based on a special quadrature formula involving derivatives at end-points. More precisely, let $\{x_j\}_{j=1}^{N-4}$ be the zeros of the Jacobi polynomial $J_{N-4}^{2,3}(x)$. We recall the quadrature rule (see [12]):

$$\int_{-1}^{1} f(x)dx \sim \sum_{j=1}^{N-4} f(x_j)\omega_j + \sum_{\mu=0}^{2} f^{(\mu)}(1)\omega_+^{(\mu)} + f(-1)\omega_-^{(0)} + f'(-1)\omega_-^{(1)},$$

where $\{\omega_j\}$ and $\{\omega_{\pm}^{(\mu)}\}$ are quadrature weights. This formula is exact for all polynomials with degree $\leqslant 2N-4$. Let $D^{(k)}$ be the kth-order differentiation matrix associated with the interior collocation points $\{x_j\}_{j=1}^{N-4}$. Then the system corresponding to the collocation scheme for (4.6) becomes

$$\left(D^{(5)}+A_1D^{(1)}+A_0\right)\mathbf{u_c}=\mathbf{f_c},$$

where $A_k = \mathrm{diag}(a_k(x_1),\ldots,a_k(x_{N-4})), k = 0,1$ and $\mathbf{u_c} = (u(x_1), \ldots, u(x_{N-4}))^T$ (likewise for $\mathbf{f_c}$).

We now make a comparison between the two methods. Let us first look at the conditioning of the systems. In Table I, we list the condition numbers of the matrices resulting from the collocation method (COL) and the generalized Jacobi spectral method (GJS).

We see that for various $a_0(x)$ and $a_1(x)$, the condition numbers of the GJS systems are all small and independent of N, while those of the COL systems increase like $O(N^{10})$.

Next, we examine the effect of roundoff errors. We take $a_0(x)=10e^{10x}$ and $a_1(x)=\sin(10x)$, and let $u(x)=\sin^3(8\pi x)$ be the exact solution of (4.6). The L^2-errors of two methods against various N are depicted in Fig. 1. We observe that the effect of roundoff errors is much more severe in the collocation method.

Finally, we emphasize that in a collocation method, the choice of the collocation points (the quadrature nodes) should be in agreement with underlying differential equations and boundary conditions. For instance, the Gauss–Lobatto points are not suitable for equations of order $\geqslant 3$ (cf. [13]). However, in a spectral-Galerkin method, the use of quadrature rules is merely to evaluate the integrals, so the usual Legendre–Gauss–Lobatto quadrature works in this case.

Table I. Condition Numbers of COL and GJS

N	Method	$a_0=0$ $a_1=0$	$a_0=10$ $a_1=0$	$a_0=50$ $a_1=1$	$a_0=100x$ $a_1=50$	$a_0=10e^{10x}$ $a_1=\sin(10x)$
16	COL	3.30E+05	3.77E+05	4.46E+05	2.49E+05	4.09E+05
16	GJS	1.00	1.07	1.42	1.62	33.05
32	COL	2.70E+08	2.78E+08	3.36E+08	1.37E+08	8.22E+08
32	GJS	1.00	1.07	1.42	1.62	33.05
64	COL	2.58E+11	2.64E+11	4.43E+11	8.11E+10	1.37E+11
64	GJS	1.00	1.07	1.42	1.62	33.05
128	COL	2.05E+14	2.10E+14	2.39E+14	1.86E+14	2.64E+14
128	GJS	1.00	1.07	1.42	1.62	33.05

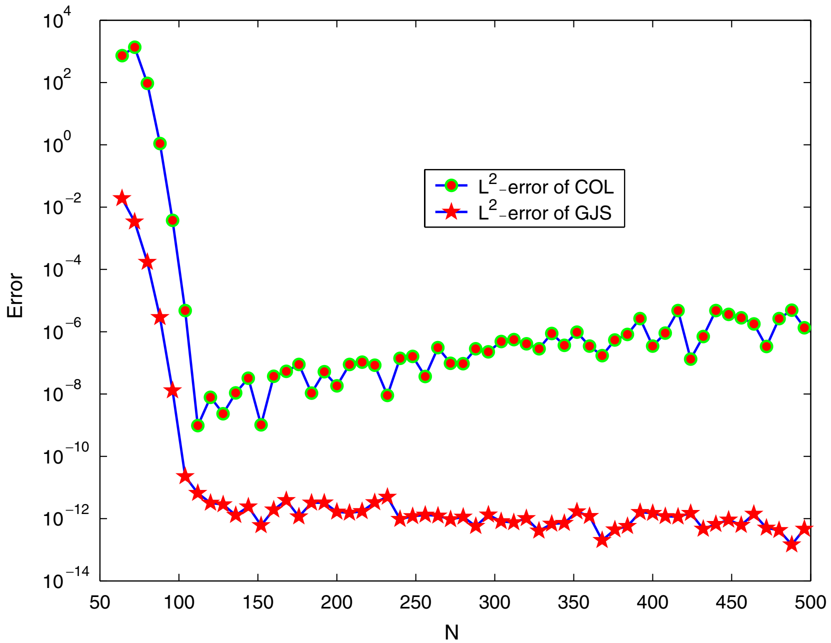

Fig. 1. L^2–errors of COL and GJS.

4.2. Application to a Fifth-Order KdV Equation

As an example of application, we consider the initial value fifth-order KdV equation:

$$\partial_t U + \gamma U \partial_x U + \nu \partial_x^3 U - \mu \partial_x^5 U = 0, \quad U(x,0) = U_0(x). \tag{4.7}$$

For $\gamma \neq 0$ and $\mu\nu > 0$, it has the following exact solution (cf. [15])

$$U(x,t) = \eta_0 + A \quad \mathrm{sech}^4\Big(\kappa(x - ct - x_0)\Big), \tag{4.8}$$

where x_0, η_0 are arbitrary constants, and

$$A = \frac{105\nu^2}{169\mu\gamma}, \quad \kappa = \sqrt{\frac{\nu}{52\mu}}, \quad c = \gamma\eta_0 + \frac{36\nu^2}{169\mu}. \tag{4.9}$$

Since $U(x,t) \to \eta_0$ exponentially as $|x| \to \infty$, we may approximate the initial value problem (4.7) by an initial boundary value problem imposed in $(-L, L)$ as long as the soliton does not reach the boundary $x = L$. Since non-homogeneous boundary conditions can be easily lifted, we only need to consider the problem (4.7) in $(-L, L)$ with the boundary conditions: $U(\pm L, t) = U'(\pm L, t) = U''(L, t) = 0$.

Setting

$$y = x/L, \quad u(y,t) = U(x,t), \quad u_0(y) = U_0(x),$$
$$\bar{\mu} = \mu/L^5, \quad \bar{\nu} = \nu/L^3, \quad \bar{\gamma} = \gamma/L,$$

the problem of interest becomes

$$\begin{aligned} &\partial_t u + \bar{\gamma} u \partial_y u + \bar{\nu} \partial_y^3 u - \bar{\mu} \partial_y^5 u = 0, \quad y \in I,\ t \in (0, T], \\ &u(y, 0) = u_0(y), \quad u(\pm 1, t) = u'(\pm 1, t) = u''(1, t) = 0. \end{aligned} \tag{4.10}$$

Let τ be the size of the time step, and define

$$t_k = k\tau, \quad v^k = v(\cdot, t_k), \quad \hat{v}^{k+1} = \frac{1}{2}\Big(v(\cdot, t_{k+1}) + v(\cdot, t_{k-1})\Big).$$

The fully discrete Crank–Nicolson leap-frog dual-Petrov-Galerkin scheme is to find $u_N^{k+1} \in V_N$ such that

$$\begin{aligned} &\frac{1}{2\tau}(u_N^{k+1} - u_N^{k-1}, v_N) + \bar{\nu}(\partial_y \hat{u}_N^{k+1}, \partial_y^2 v_N) - \bar{\mu}(\partial_y^3 \hat{u}_N^{k+1}, \partial_y^2 v_N) \\ &\quad = -\bar{\gamma}(u_N^k \partial_y u_N^k, v_N), \quad \forall v_N \in V_N^*, \end{aligned} \tag{4.11}$$

where the “dual” spaces V_N and V_N^* are defined in (3.3) with $m = 2$. Hence, at each time step, one only needs to solve the following fifth-order equation (with constant coefficients):

$$-\left(\hat{u}_N^{k+1}\right)^{(5)} + \frac{\bar{\nu}}{\bar{\mu}}\left(\hat{u}_N^{k+1}\right)^{(3)} + \frac{1}{\tau\bar{\mu}}\hat{u}_N^{k+1} = F(u_N^k, u_N^{k-1}). \tag{4.12}$$

We consider the problem (4.7)–(4.9) with

$$\mu = \gamma = 1, \quad \nu = 1.1, \quad \eta_0 = 0, \quad x_0 = -10.$$

In the computation, we take $L = 50$, $N = 120$ and $\tau = 0.001$. In Fig. 2, we plot the pointwise maximum errors against various N at $t = 1, 50, 100$. It is clear from the figure that the convergence rate behaves like e^{-cN}.

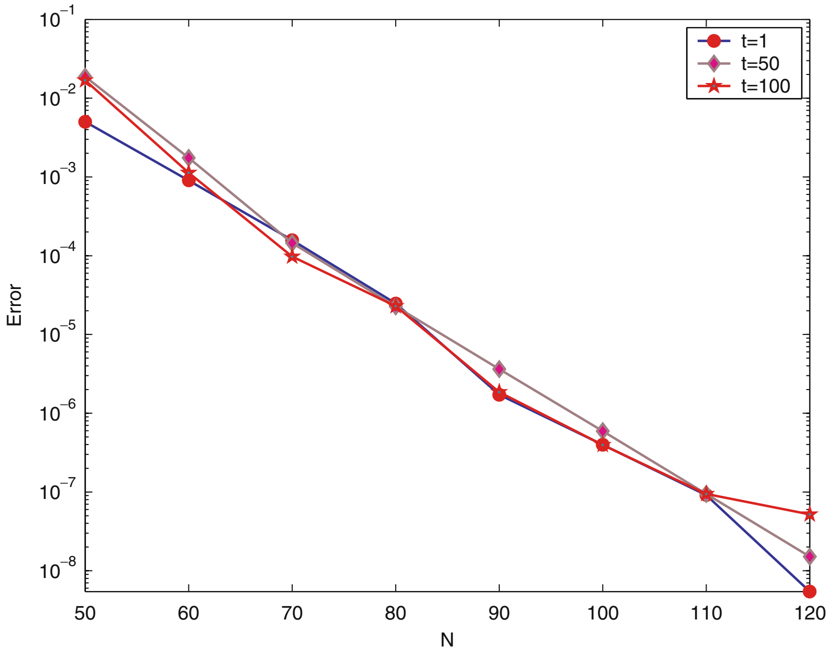

Fig. 2. Maximum error versus N at $t = 1, 50, 100$.

5. CONCLUDING REMARKS

We considered a special case (i.e., with α, β being negative integers) of the generalized Jacobi polynomials and its application to spectral-Galerkin methods. It is shown that GJPs are natural basis functions for spectral approximations of differential equations with boundary conditions that can be automatically satisfied by the corresponding GJPs. This is especially convenient for high-order differential equations for which it is difficult to design a suitable finite difference, finite element or collocation-type methods due to the many boundary conditions involved. Unlike in a collocation method for which special quadratures involving derivatives at the end points need to be developed, the implementations using GJPs are simple and straightforward. Moreover, the use of GJPs leads to much simplified analysis, more precise error estimates and well conditioned algorithms. The extension to rectangular multidimensional domains is straightforward using tensor product.

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