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Jacobi interpolation approximations and their applications to singular differential equations

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Jacobi–Gauss-type interpolations are considered. Some approximation results in certain Hilbert spaces are established. They are used for numerical solutions of singular differential equations and other related problems. The numerical results are illustrated.

Keywords: Jacobi–Gauss-type interpolations, numerical solutions of singular differential equations

AMS subject classification: 41A10, 65D05, 65L10

1. Introduction

Spectral method has high accuracy, and so has been successfully used for computations in science and engineering, see, e.g., Gottlieb and Orszag [16], Canuto et al. [8], Bernardi and Maday [5], and Guo [18]. In particular, Legendre approximation and Chebyshev approximation are widely applied to various non-singular problems in bounded domains. But in many practical cases, solutions of differential equations are singular. The singularities could be caused by several factors, such as degenerated coefficients in differential equations, unboundness of data and corners of domains. Some techniques have been proposed to overcome this trouble. For instance, Boyd [6,7] used some orthogonal polynomials to approximate solutions with endpointed weak singularities. This method gives better numerical results near the endpoints. Stenger [36] fitted singular solutions by Sinc base functions, which can reach a convergence of exponential order. Another method for solving singular problems numerically is to use Jacobi approximations. Bernardi and Maday [4] considered ultraspherical approximations in some weighted Sobolev spaces. Recently, Guo [21,22] developed Jacobi approximations in certain Hilbert spaces with their applications to singular differential equations. A related problem is numerical simulation of differential equations in unbounded domains. By some suitable variable transformations, the original problems may become some singular problems in bounded domains. Guo [19,20] used Jacobi approximation for such problems. Whereas in actual calculations, Jacobi interpolation approximations are more preferable, since we only need to evaluate the values of unknown functions at the interpolation points. The purpose of this paper is to investigate Jacobi interpolation approximations and their applications to numerical solutions of singular differential equations. In section 2, we recall some basic results related to Jacobi polynomials and Jacobi interpolations. In section 3, we derive several weighted imbedding inequalities and inverse inequalities, and discuss various orthogonal projections in certain Hilbert spaces. All results in this section will be used in the further discussions. In section 4, we give the main results of this paper. We study three kinds of Jacobi–Gauss-type interpolation approximations in specific Hilbert spaces. The corresponding results lay the mathematical foundation of Jacobi pseudospectral approximation, and play important roles in numerical analysis of Jacobi pseudospectral method for singular differential equations and related problems in unbounded domains. Section 5 is devoted to some applications. As examples, we consider a linear steady singular problem and a nonlinear singular evolutionary problem. We construct the corresponding pseudospectral schemes and prove their spectral accuracy. In section 6, we present some numerical results, which coincide with the theoretical analysis. Finally, we discuss some other applications of the Jacobi interpolation approximations.

2. Some basic results

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be certain weight function in the usual sense. For $1 \leqslant p \leqslant \infty$, set

$$L^p_\chi(\Lambda) = \big\{v \mid v \text{ is measurable and } \|v\|_{L^p_\chi} < \infty\big\},$$

equipped with the norm

$$\|v\|_{L^p_\chi} = \begin{cases} \left(\displaystyle\int_\Lambda \big|v(x)\big|^p \chi(x)\,\mathrm{d}x\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} \big|v(x)\big|, & p = \infty. \end{cases}$$

In particular, $\|v\|_\chi = \|v\|_{L^2_\chi}$ and

$$(u, v)_\chi = \int_\Lambda u(x)v(x)\chi(x)\,\mathrm{d}x, \quad \forall u, v \in L^2_\chi(\Lambda).$$

For simplicity, let $\partial_x v(x) = (\partial/\partial x)v(x)$, etc. For non-negative integer m, define

$$H^m_\chi(\Lambda) = \big\{v \mid \partial^k_x v \in L^2_\chi(\Lambda),\ 0 \leqslant k \leqslant m\big\},$$

with the semi-norm and the norm as

$$|v|_{m,\chi} = \|\partial^m_x v\|_\chi, \qquad \|v\|_{m,\chi} = \left(\sum_{k=0}^m |v|^2_{k,\chi}\right)^{1/2}.$$

For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ by space interpolation as in Adams [1]. Its semi-norm and norm are denoted by $|\cdot|_{r,\chi}$ and $\|\cdot\|_{r,\chi}$, respectively. For $\chi(x) \equiv 1$, we denote $(u, v) = (u, v)_\chi$, $\|v\| = \|v\|_\chi$ and $\|v\|_r = \|v\|_{r,\chi}$. Besides, $\|v\|_\infty = \|v\|_{L^\infty}$. Let $\mathcal{D}(\Lambda)$ be the set of all infinitely differentiable functions with compact supports in Λ, and $H^r_{0,\chi}(\Lambda)$ be its closure in $H^r_\chi(\Lambda)$.

Let $\alpha, \beta > -1$. The Jacobi polynomials are defined by

$$(1-x)^{\alpha}(1+x)^{\beta} J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l\, l!} \partial_x^l \big((1-x)^{l+\alpha}(1+x)^{l+\beta}\big), \quad l = 0, 1, 2, \ldots.$$

They are the eigenfunctions of the singular Sturm–Liouville problem

$$\partial_x \big((1-x)^{\alpha+1}(1+x)^{\beta+1} \partial_x v(x)\big) + \lambda (1-x)^{\alpha}(1+x)^{\beta} v(x) = 0, \quad x \in \Lambda, \tag{2.1}$$

with the corresponding eigenvalues $\lambda_l^{(\alpha,\beta)} = l(l+\alpha+\beta+1)$. It is noted that the coefficient of the term x^l in the expression of $J_l^{(\alpha,\beta)}(x)$ is

$$K_l^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+2l+1)}{2^l\, l!\, \Gamma(\alpha+\beta+l+1)}. \tag{2.2}$$

Besides,

$$J_l^{(\alpha,\beta)}(-x) = (-1)^l J_l^{(\beta,\alpha)}(x), \qquad J_l^{(\alpha,\beta)}(1) = \frac{\Gamma(l+\alpha+1)}{l!\, \Gamma(\alpha+1)}. \tag{2.3}$$

Let $\Gamma(x)$ be the Gamma function. We know from Askey [3] that

$$J_l^{(\alpha,\beta)}(x) = \frac{\Gamma(l+\beta+1)}{\Gamma(l+\alpha+\beta+1)} \sum_{k=0}^{l} \frac{(2k+\alpha+\beta)\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta+1)} J_k^{(\alpha-1,\beta)}(x), \tag{2.4}$$

$$J_l^{(\alpha,\beta)}(x) = \frac{\Gamma(l+\alpha+1)}{\Gamma(l+\alpha+\beta+1)} \sum_{k=0}^{l} (-1)^{l-k} \frac{(2k+\alpha+\beta)\Gamma(k+\alpha+\beta)}{\Gamma(k+\alpha+1)} J_k^{(\alpha,\beta-1)}(x). \tag{2.5}$$

The Jacobi polynomials satisfy several recurrence relations. One of them is as follows (see Askey [3]),

$$\partial_x J_l^{(\alpha,\beta)}(x) = \frac{1}{2}(l+\alpha+\beta+1) J_{l-1}^{(\alpha+1,\beta+1)}(x). \tag{2.6}$$

Another recurrence relation is stated as

$$\begin{aligned}\int_{-1}^{x} J_l^{(\alpha,\beta)}(y)\,\mathrm{d}y = {} & a_l\big(J_{l+1}^{(\alpha,\beta)}(x) - J_{l+1}^{(\alpha,\beta)}(-1)\big) + b_l\big(J_l^{(\alpha,\beta)}(x) - J_l^{(\alpha,\beta)}(-1)\big) \\ & + c_l\big(J_{l-1}^{(\alpha,\beta)}(x) - J_{l-1}^{(\alpha,\beta)}(-1)\big),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned} a_l &= \frac{2(l+\alpha+\beta+1)}{(2l+\alpha+\beta+1)(2l+\alpha+\beta+2)}, \\ b_l &= \frac{2(\alpha-\beta)}{(2l+\alpha+\beta)(2l+\alpha+\beta+2)}, \\ c_l &= \frac{-2(l+\alpha)(l+\beta)}{(l+\alpha+\beta)(2l+\alpha+\beta)(2l+\alpha+\beta+1)}. \end{aligned}$$

The derivation of (2.7) is given in appendix A.

Let $\chi^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. The set of Jacobi polynomials is the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system,

$$\left(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)}\right)_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \tag{2.8}$$

where $\delta_{l,m}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}. \tag{2.9}$$

By (2.1) and (2.8), the set $\{\partial_x J_l^{(\alpha,\beta)}(x)\}$ is the $L^2_{\chi^{(\alpha+1,\beta+1)}}(\Lambda)$-orthogonal system,

$$\left(\partial_x J_l^{(\alpha,\beta)}, \partial_x J_m^{(\alpha,\beta)}\right)_{\chi^{(\alpha+1,\beta+1)}} = \lambda_l^{(\alpha,\beta)}\gamma_l^{(\alpha,\beta)}\delta_{l,m}. \tag{2.10}$$

Now let N be any positive integer, and $\mathcal{P}_N$ be the set of all algebraic polynomials of degree at most N. The sets ${}_0\mathcal{P}_N = \{v \mid v \in \mathcal{P}_N, v(-1) = 0\}$ and $\mathcal{P}_N^0 = \{v \mid v \in \mathcal{P}_N, v(-1) = v(1) = 0\}$.

We now introduce three Jacobi–Gauss-type interpolations. To do this, let $\widetilde{\Lambda} = [-1, 1)$ and $\overline{\Lambda} = [-1, 1]$.

Jacobi–Gauss interpolation. Denote by $\zeta_{G,N,j}^{(\alpha,\beta)}$ the zeros of $J_{N+1}^{(\alpha,\beta)}(x)$, $0 \leqslant j \leqslant N$. They are arranged in decreasing order. Let $\omega_{G,N,j}^{(\alpha,\beta)}$ be the corresponding Christoffel numbers,

$$\omega_{G,N,j}^{(\alpha,\beta)} = \frac{1}{\partial_x J_{N+1}^{(\alpha,\beta)}(\zeta_{G,N,j}^{(\alpha,\beta)})}\int_{\Lambda}\frac{J_{N+1}^{(\alpha,\beta)}(x)}{x-\zeta_{G,N,j}^{(\alpha,\beta)}}\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x, \quad 0 \leqslant j \leqslant N. \tag{2.11}$$

By formulas (15.3.1) and (15.3.10) of Szegö [38], we know that

$$\omega_{G,N,j}^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(N+\alpha+2)\Gamma(N+\beta+2)}{\Gamma(N+2)\Gamma(N+\alpha+\beta+2)}\left(1-\left(\zeta_{G,N,j}^{(\alpha,\beta)}\right)^2\right)^{-1}\left(\partial_x J_{N+1}^{(\alpha,\beta)}\left(\zeta_{G,N,j}^{(\alpha,\beta)}\right)\right)^{-2} \tag{2.12}$$

and

$$\omega_{G,N,j}^{(\alpha,\beta)} \sim \frac{2^{\alpha+\beta+1}\pi}{N+1}\left(1-\zeta_{G,N,j}^{(\alpha,\beta)}\right)^{\alpha+1/2}\left(1+\zeta_{G,N,j}^{(\alpha,\beta)}\right)^{\beta+1/2}. \tag{2.13}$$

As pointed out in Szegö [38], for any $\phi \in \mathcal{P}_{2N+1}$,

$$\int_{\Lambda}\phi(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{j=0}^{N}\phi\left(\zeta_{G,N,j}^{(\alpha,\beta)}\right)\omega_{G,N,j}^{(\alpha,\beta)}. \tag{2.14}$$

For any $v \in C(\Lambda)$, the Jacobi–Gauss interpolant $\mathcal{I}_{G,N,\alpha,\beta}v(x) \in \mathcal{P}_N$, satisfying

$$\mathcal{I}_{G,N,\alpha,\beta}v\left(\zeta_{G,N,j}^{(\alpha,\beta)}\right) = v\left(\zeta_{G,N,j}^{(\alpha,\beta)}\right), \quad 0 \leqslant j \leqslant N.$$

Jacobi–Gauss–Radau interpolation. Let $\zeta^{(\alpha,\beta)}_{R,N,N} = -1$, and $\zeta^{(\alpha,\beta)}_{R,N,j}$, $0 \leqslant j \leqslant N-1$, be the zeros of $J^{(\alpha,\beta+1)}_N(x)$. They are arranged in decreasing order. Set $R^{(\alpha,\beta)}_{l+1}(x) = (1+x)J^{(\alpha,\beta+1)}_l(x)$. Furthermore, let $\omega^{(\alpha,\beta)}_{R,N,j}$ be the corresponding Christoffel numbers,

$$\omega^{(\alpha,\beta)}_{R,N,j} = \frac{1}{\partial_x R^{(\alpha,\beta)}_{N+1}(\zeta^{(\alpha,\beta)}_{R,N,j})} \int_\Lambda \frac{R^{(\alpha,\beta)}_{N+1}(x)}{x - \zeta^{(\alpha,\beta)}_{R,N,j}} \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x, \quad 0 \leqslant j \leqslant N. \tag{2.15}$$

Clearly,

$$\zeta^{(\alpha,\beta)}_{R,N,j} = \zeta^{(\alpha,\beta+1)}_{G,N-1,j}, \quad 0 \leqslant j \leqslant N-1. \tag{2.16}$$

We obtain from (2.11), (2.15) and (2.16) that

$$\omega^{(\alpha,\beta)}_{R,N,j} = \big(1 + \zeta^{(\alpha,\beta+1)}_{G,N-1,j}\big)^{-1} \omega^{(\alpha,\beta+1)}_{G,N-1,j}, \quad 0 \leqslant j \leqslant N-1. \tag{2.17}$$

Therefore, by (2.13), (2.16) and (2.17),

$$\omega^{(\alpha,\beta)}_{R,N,j} \sim \frac{2^{\alpha+\beta+2}\pi}{N} \big(1 - \zeta^{(\alpha,\beta)}_{R,N,j}\big)^{\alpha+1/2} \big(1 + \zeta^{(\alpha,\beta)}_{R,N,j}\big)^{\beta+1/2}, \quad 0 \leqslant j \leqslant N-1. \tag{2.18}$$

Moreover, for any $\phi \in \mathcal{P}_{2N}$,

$$\int_\Lambda \phi(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{j=0}^{N} \phi\big(\zeta^{(\alpha,\beta)}_{L,N,j}\big)\omega^{(\alpha,\beta)}_{L,N,j}. \tag{2.19}$$

For any $v \in C(\widetilde{\Lambda})$, the Jacobi–Gauss–Radau interpolant $\mathcal{I}_{R,N,\alpha,\beta}v(x) \in \mathcal{P}_N$, satisfying

$$\mathcal{I}_{R,N,\alpha,\beta}v\big(\zeta^{(\alpha,\beta)}_{R,N,j}\big) = v\big(\zeta^{(\alpha,\beta)}_{R,N,j}\big), \quad 0 \leqslant j \leqslant N.$$

Jacobi–Gauss–Lobatto interpolation. Let $\zeta^{(\alpha,\beta)}_{L,N,0} = 1$, $\zeta^{(\alpha,\beta)}_{L,N,N} = -1$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$, $1 \leqslant j \leqslant N-1$, be the zeros of $\partial_x J^{(\alpha,\beta)}_N(x)$. They are arranged in decreasing order. Set $L^{(\alpha,\beta)}_{N+1}(x) = (1-x^2)\partial_x J^{(\alpha,\beta)}_N(x)$, and $\omega^{(\alpha,\beta)}_{L,N,j}$ be the corresponding Christoffel numbers,

$$\omega^{(\alpha,\beta)}_{L,N,j} = \frac{1}{\partial_x L^{(\alpha,\beta)}_{N+1}(\zeta^{(\alpha,\beta)}_{L,N,j})} \int_\Lambda \frac{L^{(\alpha,\beta)}_{N+1}(x)}{x - \zeta^{(\alpha,\beta)}_{L,N,j}} \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x, \quad 0 \leqslant j \leqslant N. \tag{2.20}$$

Clearly, by (2.6),

$$\zeta^{(\alpha,\beta)}_{L,N+2,j} = \zeta^{(\alpha+1,\beta+1)}_{G,N,j-1}, \quad 1 \leqslant j \leqslant N+1. \tag{2.21}$$

It can be also verified from (2.11), (2.20) and (2.21) that

$$\omega^{(\alpha,\beta)}_{L,N+2,j} = \big(1 - \big(\zeta^{(\alpha+1,\beta+1)}_{G,N,j-1}\big)^2\big)^{-1} \omega^{(\alpha+1,\beta+1)}_{G,N,j-1}, \quad 1 \leqslant j \leqslant N+1. \tag{2.22}$$

This fact with (2.13) and (2.21) implies that

$$\omega^{(\alpha,\beta)}_{L,N,j} \sim \frac{c}{N} \big(1 - \zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\alpha+1/2} \big(1 + \zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\beta+1/2}. \tag{2.23}$$

Moreover, for any $\phi \in \mathcal{P}_{2N-1}$,

$$\int_\Lambda \phi(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{j=0}^{N} \phi\big(\zeta_{L,N,j}^{(\alpha,\beta)}\big)\omega_{L,N,j}^{(\alpha,\beta)}. \tag{2.24}$$

For any $v \in C(\overline{\Lambda})$, the Jacobi–Gauss–Lobatto interpolant $\mathcal{I}_{L,N,\alpha,\beta}v(x) \in \mathcal{P}_N$, satisfying

$$\mathcal{I}_{L,N,\alpha,\beta}v\big(\zeta_{L,N,j}^{(\alpha,\beta)}\big) = v\big(\zeta_{L,N,j}^{(\alpha,\beta)}\big), \quad 0 \leqslant j \leqslant N.$$

We next introduce the discrete norm $\|v\|_{L^p_{\chi^{(\alpha,\beta)}},Z,N}$ associated with the interpolation points $\zeta_{Z,N,j}^{(\alpha,\beta)}$ and the weights $\omega_{Z,N,j}^{(\alpha,\beta)}$, namely,

$$\|v\|_{L^p_{\chi^{(\alpha,\beta)}},Z,N} = \begin{cases} \left(\displaystyle\sum_{j=0}^{N} \big|v\big(\zeta_{Z,N,j}^{(\alpha,\beta)}\big)\big|^p \omega_{Z,N,j}^{(\alpha,\beta)}\right)^{1/p}, & \text{for } 1 \leqslant p < \infty, \\ \displaystyle\max_{0\leqslant j\leqslant N} \big|v\big(\zeta_{Z,N,j}^{(\alpha,\beta)}\big)\big|, & \text{for } p = \infty, \end{cases}$$

where $Z = G$, R and L, respectively. In particular, the discrete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ inner product and the corresponding discrete norm are defined by

$$(u,v)_{\chi^{(\alpha,\beta)},Z,N} = \sum_{j=0}^{N} u\big(\zeta_{Z,N,j}^{(\alpha,\beta)}\big)v\big(\zeta_{Z,N,j}^{(\alpha,\beta)}\big)\omega_{Z,N,j}^{(\alpha,\beta)}, \qquad \|v\|_{\chi^{(\alpha,\beta)},Z,N} = (v,v)^{1/2}_{\chi^{(\alpha,\beta)},Z,N}.$$

By (2.14), (2.19) and (2.24),

$$(\phi,\psi)_{\chi^{(\alpha,\beta)},Z,N} = (\phi,\psi)_{\chi^{(\alpha,\beta)}}, \quad \forall \phi.\psi \in \mathcal{P}_{2N+\lambda} \tag{2.25}$$

where $\lambda = 1$, 0 and -1 for $Z = G$, R and L, respectively. Besides, for any $\phi \in \mathcal{P}_N$ (see appendix B),

$$\|\phi\|_{\chi^{(\alpha,\beta)}} \leqslant \|\phi\|_{\chi^{(\alpha,\beta)},L,N} \leqslant \sqrt{2 + \frac{\alpha+\beta+1}{N}}\,\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{2.26}$$

Generally, we have the following result.

Lemma 2.1. Let $\phi \in \mathcal{P}_N$ and $1 \leqslant p < \infty$. In addition, $\phi(-1) = 0$ for $Z = R$, and $\phi(-1) = \phi(1) = 0$ for $Z = L$. Then

$$\|\phi\|_{L^p_{\chi^{(\alpha,\beta)}},Z,N} \leqslant c(p)\|\phi\|_{L^p_{\chi^{(\alpha,\beta)}}}, \quad Z = G, R, L, \tag{2.27}$$

where $c(p)$ is a positive constant depending on p.

Proof. Let $\mu(x)$ be a Jacobi weight. $\xi^{(j)}$ and $\rho^{(j)}$, $1 \leqslant j \leqslant M$, are the nodes and weights of the Jacobi–Gauss quadrature associated with $\mu(x)$. Further, let m be a positive integer. Nevai [33] (also see Szabados and Vèrtesi [39]) proved that for any $\psi \in \mathcal{P}_{mM}$, any Jacobi weight $f(x)$ and $0 < p < \infty$,

$$\sum_{j=1}^{M} \left|\psi\left(\xi^{(j)}\right)\right|^p f\left(\xi^{(j)}\right)\rho^{(j)} \leqslant c(p) \int_\Lambda \left|\psi(x)\right|^p f(x)\mu(x)\,\mathrm{d}x. \tag{2.28}$$

Taking $\psi(x) = \phi(x)$, $f(x) = 1$ and $\mu(x) = \chi^{(\alpha,\beta)}(x)$ in (2.28), we obtain (2.27) with $Z = G$, immediately. We next take $f(x) = (1+x)^{-1}$ and $\mu(x) = \chi^{(\alpha,\beta+1)}(x)$. Then by (2.16), (2.17) and (2.28),

$$\begin{aligned}
\|\phi\|^p_{L^p_{\chi^{(\alpha,\beta)}},R,N} &= \sum_{j=0}^{N} \left|\phi\left(\zeta^{(\alpha,\beta)}_{R,N,j}\right)\right|^p \omega^{(\alpha,\beta)}_{R,N,j} = \sum_{j=0}^{N-1} \left|\phi\left(\zeta^{(\alpha,\beta)}_{R,N,j}\right)\right|^p \omega^{(\alpha,\beta)}_{R,N,j} \\
&= \sum_{j=0}^{N-1} \left|\phi\left(\zeta^{(\alpha,\beta+1)}_{G,N-1,j}\right)\right|^p \left(1+\zeta^{(\alpha,\beta+1)}_{G,N-1,j}\right)^{-1} \omega^{(\alpha,\beta+1)}_{G,N-1,j} \\
&\leqslant c(p) \int_\Lambda \left|\phi(x)\right|^p (1+x)^{-1}\chi^{(\alpha,\beta+1)}(x)\,\mathrm{d}x \\
&= c(p)\|\phi\|^p_{L^p_{\chi^{(\alpha,\beta)}}}.
\end{aligned}$$

This leads to (2.27) with $Z = R$. Similarly, taking $f(x) = (1-x^2)^{-1}$ and $\mu(x) = \chi^{(\alpha+1,\beta+1)}(x)$ in (2.28), we derive (2.27) with $Z = L$ by using (2.21), (2.22) and a similar argument as in the above. $\square$

3. Jacobi orthogonal projections

In order to derive the main results, we need some preparations. They include some weighted imbedding inequalities, inverse inequalities and various orthogonal projections.

Let $\mathbb{N}$ be the set of all non-negative integers. Denote by c a generic positive constant, independent of any function and N. But it might depend on real numbers α, β, γ or δ. Without further mention, we always assume that $\alpha, \beta, \gamma, \delta > -1$.

We first give some inverse inequalities, see lemmas 3.1–3.3.

Lemma 3.1 (Theorem 2.1 of Guo [22]). For any $\phi \in \mathcal{P}_N$ and $1 \leqslant p \leqslant q \leqslant \infty$,

$$\|\phi\|_{L^q_{\chi^{(\alpha,\beta)}}} \leqslant cN^{\sigma(\alpha,\beta)(1/p-1/q)}\|\phi\|_{L^p_{\chi^{(\alpha,\beta)}}}, \tag{3.1}$$

where

$$\sigma(\alpha,\beta) = \begin{cases} 2\max(\alpha,\beta)+2, & \text{if } \max(\alpha,\beta) \geqslant -\dfrac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 3.2 (Theorem 2.2 of Guo [22]). For any $\phi \in \mathcal{P}_N$ and $r \geqslant 0$,

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \leqslant cN^{2r}\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{3.2}$$

If, in addition, $\alpha, \beta > r-1$, then

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \leqslant cN^{r}\|\phi\|_{\chi^{(\alpha-r,\beta-r)}}. \tag{3.3}$$

Lemma 3.3. For any $\phi \in \mathcal{P}_N^0$,

$$\|\partial_x\phi\|_{\chi^{(\alpha,\beta)}} \leqslant cN\|\phi\|_{\chi^{(\alpha-1,\beta-1)}}. \tag{3.4}$$

Proof. Let

$$L_l^{(\alpha,\beta)}(x) = \big(1-x^2\big)\partial_x J_{l-1}^{(\alpha,\beta)}(x), \quad l = 2,3,\ldots.$$

By (2.10),

$$\big(L_l^{(\alpha,\beta)}, L_m^{(\alpha,\beta)}\big)_{\chi^{(\alpha-1,\beta-1)}} = \lambda_{l-1}^{(\alpha,\beta)}\gamma_{l-1}^{(\alpha,\beta)}\delta_{l,m}.$$

Thus the set of $\{L_l^{(\alpha,\beta)}(x)\}$ is an $L^2_{\chi^{(\alpha-1,\beta-1)}}(\Lambda)$-orthogonal system. Moreover, $L_l^{(\alpha,\beta)}(x)$, $2 \leqslant l \leqslant N$, form a basis of $\mathcal{P}_N^0$. Therefore, for any $\phi \in \mathcal{P}_N^0$,

$$\phi(x) = \sum_{l=2}^{N}\widehat{\phi}_l^{\,*}L_l^{(\alpha,\beta)}(x)$$

and

$$\|\phi\|^2_{\chi^{(\alpha-1,\beta-1)}} = \sum_{l=2}^{N}\lambda_{l-1}^{(\alpha,\beta)}\gamma_{l-1}^{(\alpha,\beta)}\big(\widehat{\phi}_l^{\,*}\big)^2.$$

Furthermore, by (2.1),

$$\big(1-x^2\big)\partial_x^2 J_l^{(\alpha,\beta)}(x) = \big((\alpha-\beta)+(\alpha+\beta+2)x\big)\partial_x J_l^{(\alpha,\beta)}(x) - \lambda_l^{(\alpha,\beta)}J_l^{(\alpha,\beta)}(x).$$

This fact and (2.6) imply that

$$\partial_x L_l^{(\alpha,\beta)}(x) = \frac{1}{2}\big((\alpha-\beta)+(\alpha+\beta)x\big)(l+\alpha+\beta)J_{l-2}^{(\alpha+1,\beta+1)}(x) - \lambda_{l-1}^{(\alpha,\beta)}J_{l-1}^{(\alpha,\beta)}(x).$$

Thus,

$$\partial_x\phi(x) = \frac{1}{2}\big((\alpha-\beta)+(\alpha+\beta)x\big)W(N,\alpha,\beta,x) - \sum_{l=2}^{N}\widehat{\phi}_l^{\,*}\lambda_{l-1}^{(\alpha,\beta)}J_{l-1}^{(\alpha,\beta)}(x),$$

where

$$W(N,\alpha,\beta,x) = \sum_{l=0}^{N-2}(l+\alpha+\beta+2)\widehat{\phi}_{l+2}^{\,*}J_l^{(\alpha+1,\beta+1)}(x).$$

For simplicity of statements, we use the following notations,

$$E_l = \frac{\Gamma(l+\beta+2)}{\Gamma(l+\alpha+\beta+2)}, \qquad F_k = \frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\beta+2)},$$

$$G_k = \frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)}, \quad H_j = \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\alpha+1)},$$

$$\psi_{j,l} = \sum_{k=j}^{l} (-1)^k G_k.$$

By virtue of (2.4), (2.5) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we obtain that

$$\begin{aligned} W(N,\alpha,\beta,x) &= \sum_{l=0}^{N-2} E_l \widehat{\phi}_{l+2}^{*} \left(\sum_{k=0}^{l} F_k J_k^{(\alpha,\beta+1)}(x) \right) \\ &= \sum_{l=0}^{N-2} E_l \widehat{\phi}_{l+2}^{*} \left(\sum_{k=0}^{l} (-1)^k G_k \left(\sum_{j=0}^{k} (-1)^j H_j J_j^{(\alpha,\beta)}(x) \right) \right) \\ &= \sum_{l=0}^{N-2} E_l \widehat{\phi}_{l+2}^{*} \left(\sum_{j=0}^{l} (-1)^j \psi_{j,l} H_j J_j^{(\alpha,\beta)}(x) \right) \\ &= \sum_{j=0}^{N-2} (-1)^j H_j J_j^{(\alpha,\beta)}(x) \left(\sum_{l=j}^{N-2} E_l \psi_{j,l} \widehat{\phi}_{l+2}^{*} \right). \end{aligned}$$

Let

$$A_k = \frac{(2k+\alpha+\beta+3)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+3)}.$$

It can be checked that

$$G_{k+1} - G_k = (\alpha-\beta) A_k.$$

By the Stirling formula (see Courant and Hilbert [12]),

$$\Gamma(s+1) = \sqrt{2\pi s}\, s^s \mathrm{e}^{-s} \big(1 + \mathcal{O}\big(s^{-1/5}\big)\big), \tag{3.5}$$

hence

$$|A_k| \leqslant c(k+1)^{\alpha-\beta-1}, \qquad |G_k| \leqslant c(k+1)^{\alpha-\beta},$$

$$|\psi_{j,l}| \leqslant c(l+1)^{\alpha-\beta}, \qquad E_l \leqslant c(l+1)^{-\alpha}, \qquad H_j \leqslant c(j+1)^{\beta+1}.$$

The above estimates with (2.8) lead to that

$$
\begin{aligned}
&\left\| W(N,\alpha,\beta,x)\right\|^2_{\chi^{(\alpha,\beta)}}\\
&\quad\leqslant c\sum_{j=0}^{N-2} H_j^2\gamma_j^{(\alpha,\beta)}\left(\sum_{l=j}^{N-2} E_l\psi_{j,l}\widehat{\phi}^*_{l+2}\right)^2\\
&\quad\leqslant c\sum_{j=0}^{N-2} H_j^2\gamma_j^{(\alpha,\beta)}\left(\sum_{l=j}^{N-2} E_l^2\psi_{j,l}^2\big(\gamma_j^{(\alpha,\beta)}\lambda_j^{(\alpha,\beta)}\big)^{-1}\right)\left(\sum_{l=j}^{N-2}\gamma_j^{(\alpha,\beta)}\lambda_j^{(\alpha,\beta)}\big(\widehat{\phi}^*_{l+2}\big)^2\right)\\
&\quad\leqslant c\sum_{j=0}^{N-2}(j+1)^{2\beta+2}\left(\sum_{l=j}^{N-2}(l+1)^{-2\beta-2}\right)\|\phi\|^2_{\chi^{(\alpha-1,\beta-1)}}\\
&\quad\leqslant cN^2\|\phi\|^2_{\chi^{(\alpha-1,\beta-1)}}.
\end{aligned}
$$

Thus,

$$
\|\partial_x\phi\|^2_{\chi^{(\alpha,\beta)}}\leqslant c\big(N^2\|\phi\|^2_{\chi^{(\alpha-1,\beta-1)}}+\|W(N,\alpha,\beta,x)\|^2_{\chi^{(\alpha,\beta)}}\big)\leqslant cN^2\|\phi\|^2_{\chi^{(\alpha-1,\beta-1)}}
$$

which completes the proof. □

We next establish some imbedding inequalities, see lemmas 3.4–3.8.

Lemma 3.4. If

$$
1<\alpha\leqslant\gamma+2,\qquad 1<\beta\leqslant\delta+2, \tag{3.6}
$$

then for any $v\in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$
\|v\|_{\chi^{(\gamma,\delta)}}\leqslant c\|v\|_{1,\chi^{(\alpha,\beta)}}. \tag{3.7}
$$

Moreover, for any $v\in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ with $v(x_0)=0,\ x_0\in\Lambda$,

$$
\|v\|_{\chi^{(\gamma,\delta)}}\leqslant c|v|_{1,\chi^{(\alpha,\beta)}}, \tag{3.8}
$$

provided that

$$
\alpha\leqslant\gamma+2,\qquad \beta\leqslant\delta+2. \tag{3.9}
$$

Proof. By a similar argument as in the proof of lemma 2.3 of Guo [22], we can get the result (3.8) with (3.9). We now prove (3.7) with (3.6). Let $\Lambda_1=(0,1)$ and $\Lambda_2=(-1,0]$. Following the same lines as in the proof of formula (13.5) of Bernardi and Maday [5], we have that for $\alpha>1$,

$$
\int_{\Lambda_1} v^2(x)\big(1-x^2\big)^{\alpha-2}\,\mathrm{d}x\leqslant\frac{\max\{\alpha^2,2\}}{(\alpha-1)^2}\int_{\Lambda_1}\big(\big(\partial_x v(x)\big)^2+v^2(x)\big)\big(1-x^2\big)^{\alpha}\,\mathrm{d}x,
$$

provided that the integral at the right side of the above formula exists. Hence

$$
\int_{\Lambda_1} v^2(x)\chi^{(\alpha-2,\beta-2)}(x)\,\mathrm{d}x\leqslant c\int_{\Lambda_1}\big(\big(\partial_x v(x)\big)^2+v^2(x)\big)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x.
$$

A similar result on Λ_2 exists for $\beta > 1$. Finally, a combination of (3.6) and the above inequalities leads to (3.7). $\square$

Lemma 3.5. If $\alpha > -1$ and $-1 < \beta < 1$ then for any $v \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$, v is continuous on any subinterval $\Lambda^* = [-1, a] \subset \overline{\Lambda}$ with $a < 1$, and $\max_{x\in\Lambda^*} |v(x)| \leqslant c\|v\|_{1,\chi^{(\alpha,\beta)}}$. If, in addition, $\alpha < 1$, then these results can be extended to $\overline{\Lambda}$.

Proof. For any $x_1, x_2 \in \Lambda^*$, we have that

$$\left|v(x_2) - v(x_1)\right| \leqslant \int_{x_1}^{x_2} \left|\partial_x v(x)\right| \mathrm{d}x \leqslant c\|\partial_x v\|_{\chi^{(\alpha,\beta)}} \left(\int_{x_1}^{x_2} \chi^{(-\alpha,-\beta)}(x)\,\mathrm{d}x\right)^{1/2}.$$

So $v(x_1) \to v(x_2)$ as $x_1 \to x_2$, i.e., $v \in C(\Lambda^*)$. Next, let $|v(x^*)| = \min_{x\in\Lambda^*} |v(x)|$. Then

$$\left|v(x)\right| - \left|v(x^*)\right| \leqslant c\|\partial_x v\|_{\chi^{(\alpha,\beta)}}.$$

Moreover,

$$\left|v(x^*)\right| \leqslant \frac{1}{a+1}\int_{-1}^{a} \left|v(x)\right| \mathrm{d}x \leqslant c\|v\|_{\chi^{(\alpha,\beta)}}$$

which leads to the first desired result. Furthermore, if $\alpha < 1$, then we can take $a = 1$ and so complete the proof. $\square$

Next, let

$${}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda) = \left\{v \mid v \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda),\ v(-1) = 0\right\}.$$

Lemma 3.6. If one of the following conditions holds,

(i) $\alpha \leqslant \gamma + 2,\ \beta \leqslant 0,\ \delta \geqslant 0$,

(ii) $\alpha \leqslant \gamma + 1,\ \beta \leqslant \delta + 2,\ 0 < \alpha < 1,\ \beta < 1$,

then for any $v \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\|v\|_{\chi^{(\gamma,\delta)}} \leqslant c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{3.10}$$

Proof. The result (3.10) with condition (i) follows directly from lemma 2.4 of Guo [22]. Now, let condition (ii) holds. Denote $\Lambda_1 = (0, 1)$ and $\Lambda_2 = (-1, 0]$. By the Hardy inequality (see Hardy et al. [24]), we know that for any measurable function ϕ and real number $\alpha < 1$,

$$\int_{\Lambda_1} \left(\frac{1}{1-x}\int_x^1 \phi(y)\,\mathrm{d}y\right)^2 (1-x)^{\alpha}\,\mathrm{d}x \leqslant \frac{4}{1-\alpha}\int_{\Lambda_1} \phi^2(x)(1-x)^{\alpha}\,\mathrm{d}x. \tag{3.11}$$

For any $v \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$, we have from lemma 3.5 that $v(x)$ is meaningful at $x = 1$. Taking $\phi(x) = \partial_x v(x)$ in (3.11), we get that

$$\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 (1-x)^{\alpha-2}\,\mathrm{d}x \leqslant \frac{4}{1-\alpha} \int_{\Lambda_1} \big(\partial_x v(x)\big)^2 (1-x)^{\alpha}\,\mathrm{d}x. \tag{3.12}$$

Let $|v(x^*)| = \max_{x\in\overline{\Lambda}} |v(x)|$. Since $v(-1) = 0$, it can be verified that for $\alpha, \beta < 1$,

$$\big|v(1)\big| \leqslant \big|v(x^*)\big| \leqslant \int_{-1}^{x^*} \big|\partial_x v(x)\big|\,\mathrm{d}x \leqslant c\|\partial_x v\|_{\chi^{(\alpha,\beta)}}. \tag{3.13}$$

Furthermore, for $\alpha > 0$,

$$\begin{aligned} &\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 (1-x)^{\alpha-1}\,\mathrm{d}x \\ &\quad \geqslant \int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,\mathrm{d}x + \int_{\Lambda_1} \big(v^2(1) - 2\big|v(1)\big|\big|v(x)\big|\big)(1-x)^{\alpha-1}\,\mathrm{d}x \\ &\quad \geqslant \int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,\mathrm{d}x + c\big(v^2(1) - 2\big|v\big(x^*\big)\big|\big|v(1)\big|\big). \end{aligned} \tag{3.14}$$

So we have from (3.12)–(3.14) that

$$\begin{aligned} &\int_{\Lambda_1} v^2(x)\chi^{(\alpha-1,\beta-2)}(x)\,\mathrm{d}x \\ &\quad \leqslant c\int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,\mathrm{d}x \\ &\quad \leqslant c\bigg(\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 (1-x)^{\alpha-1}\,\mathrm{d}x + \big|v\big(x^*\big)\big|\big|v(1)\big|\bigg) \\ &\quad \leqslant c\bigg(\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 (1-x)^{\alpha-2}\,\mathrm{d}x + \|\partial_x v\|^2_{\chi^{(\alpha,\beta)}}\bigg) \leqslant c\|\partial_x v\|^2_{\chi^{(\alpha,\beta)}}. \end{aligned} \tag{3.15}$$

On the other hand, $v(-1) = 0$ and so by the Hardy inequality,

$$\int_{\Lambda_2} v^2(x)(1+x)^{\beta-2}\,\mathrm{d}x \leqslant c\int_{\Lambda_2} \big(\partial_x v(x)\big)^2 (1+x)^{\beta}\,\mathrm{d}x.$$

Thus

$$\int_{\Lambda_2} v^2(x)\chi^{(\alpha-1,\beta-2)}(x)\,\mathrm{d}x \leqslant c\int_{\Lambda_2} \big(\partial_x v(x)\big)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x.$$

Finally, we can get the desired result by condition (ii) and the above statements. □

Remark 3.1. If $\alpha > -1,\ \beta = 0$ or $-1 < \alpha, \beta < 1$, then the semi-norm $|\cdot|_{1,\chi^{(\alpha,\beta)}}$ is a norm of the space ${}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$, which is equivalent to the norm $\|\cdot\|_{1,\chi^{(\alpha,\beta)}}$. Indeed, taking

$\alpha = \gamma$ and $\beta = \delta$ in (3.10), we can get the desired result with $\alpha > -1$ and $\beta = 0$. Next, by (3.12),

$$\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \leqslant c \int_{\Lambda_1} \big(v(x) - v(1)\big)^2 (1-x)^{\alpha-2}\,\mathrm{d}x$$
$$\leqslant \frac{4}{1-\alpha} \int_{\Lambda_1} \big(\partial_x v(x)\big)^2 (1-x)^{\alpha}\,\mathrm{d}x.$$

Moreover, by a similar argument as in the derivation of (3.14),

$$\int_{\Lambda_1} \big(v(x) - v(1)\big)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \geqslant \int_{\Lambda_1} v^2(x) \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x + c\big(v^2(1) - 2\big|v\big(x^*\big)\big|\big|v(1)\big|\big).$$

We can also estimate the left side of the above inequality as (3.15). Therefore

$$\int_{\Lambda_1} v^2(x) \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \leqslant c\bigg(\int_{\Lambda_1} \big(\partial_x v(x)\big)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x + \big|v\big(x^*\big)\big|\big|v(1)\big|\bigg) \leqslant c\|\partial_x v\|^2_{\chi^{(\alpha,\beta)}}.$$

A similar inequality is valid on Λ_2. A combination of them leads to the desired result with $-1 < \alpha, \beta < 1$.

Lemma 3.7. If one of the following conditions holds,

(i) $\alpha \leqslant \gamma + 2,\ \beta \leqslant 0,\ \delta \geqslant 0$,

(ii) $\alpha \leqslant 0,\ \beta \leqslant \delta + 2,\ \gamma \geqslant 0$,

(iii) $\alpha \leqslant \gamma + 2,\ \beta \leqslant \delta + 2,\ -1 < \alpha, \beta < 1$,

then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$,

$$\|v\|_{\chi^{(\gamma,\delta)}} \leqslant c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{3.16}$$

Proof. The validity of (3.16) with condition (i) or condition (ii) is ensured by lemma 2.4 and remark 2.3 of Guo [22]. Now let condition (iii) hold. Set $\Lambda_1 = (0,1)$ and $\Lambda_2 = (-1,0]$. Taking $\phi(x) = \partial_x v(x)$ in (3.11), we get that

$$\int_{\Lambda_1} v^2(x) \chi^{(\alpha-2,\beta-2)}(x)\,\mathrm{d}x \leqslant c \int_{\Lambda_1} \big(\partial_x v(x)\big)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x.$$

A similar inequality is valid on Λ_2. Then (3.16) follows. □

Remark 3.2. As a result of lemma 3.7, we assert that if one of the conditions holds: (i) $\alpha > -1$, $\beta = 0$, (ii) $\alpha = 0$, $\beta > -1$, (iii) $-1 < \alpha, \beta < 1$, then the semi-norm $|\cdot|_{1,\chi^{(\alpha,\beta)}}$ is a norm of the space $H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$, which is equivalent to the norm $\|\cdot\|_{1,\chi^{(\alpha,\beta)}}$.

Lemma 3.8. For $-1 < \alpha, \beta < 1$, the mapping $\mathcal{L}^{\alpha,\beta} : v \to v\chi^{(\alpha,\beta)}$ is an isomorphism from $H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ onto $H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$.

Proof. For any $v \in \mathcal{D}(\Lambda)$, we get from (3.16) with condition (iii) of lemma 3.7 that

$$\begin{aligned}
&\left\|\partial_x\left(v\chi^{(\alpha,\beta)}\right)\right\|^2_{\chi^{(-\alpha,-\beta)}} \\
&\leqslant \int_\Lambda \left(\partial_x v(x)\right)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x + 4\left(\alpha^2+\beta^2\right)\int_\Lambda v^2(x)\chi^{(\alpha-2,\beta-2)}(x)\,\mathrm{d}x \\
&\leqslant c\|\partial_x v\|^2_{\chi^{(\alpha,\beta)}}.
\end{aligned}$$

Clearly,

$$\|v\chi^{(\alpha,\beta)}\|_{\chi^{(-\alpha,-\beta)}} = \|v\|_{\chi^{(\alpha,\beta)}}.$$

The previous estimates imply that

$$\left\|v\chi^{(\alpha,\beta)}\right\|_{1,\chi^{(-\alpha,-\beta)}} \leqslant c\|v\|_{1,\chi^{(\alpha,\beta)}}.$$

So $\mathcal{L}^{\alpha,\beta}$ is a linear continuous mapping from $\mathcal{D}(\Lambda)$ provided with the norm $\|\cdot\|_{1,\chi^{(\alpha,\beta)}}$ into $\mathcal{D}(\Lambda)$ provided with the norm $\|\cdot\|_{1,\chi^{(-\alpha,-\beta)}}$. Conversely, we can use (3.16) with condition (iii) of lemma 3.7 to show that the inverse mapping $\mathcal{L}^{-\alpha,-\beta} : v \to v\chi^{(-\alpha,-\beta)}$ is also a linear continuous mapping from $\mathcal{D}(\Lambda)$ provided with the norm $\|\cdot\|_{1,\chi^{(-\alpha,-\beta)}}$ into $\mathcal{D}(\Lambda)$ provided with the norm $\|\cdot\|_{1,\chi^{(\alpha,\beta)}}$. Finally, a density argument leads to the conclusion. $\square$

We now consider various orthogonal projections. For technical reasons, Guo [22] introduced the Hilbert space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$. For any $r \in \mathbb{N}$,

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \left\{v \mid v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},A} < \infty\right\},$$

where

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{[(r-1)/2]} \left\|\left(1-x^2\right)^{r/2-k}\partial_x^{r-k}v\right\|^2_{\chi^{(\alpha,\beta)}} + \|v\|^2_{[r/2],\chi^{(\alpha,\beta)}}\right)^{1/2}.$$

In particular, $H^0_{\chi^{(\alpha,\beta)},A}(\Lambda) = L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$. For any real $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation. Guo [22] also introduced other spaces. For any $\mu \in \mathbb{N}$,

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda) = \left\{v \mid \partial_x^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},A}(\Lambda)\right\}$$

with the norm

$$\|v\|_{r,\chi^{(\alpha,\beta)},*,\mu} = \left\|\partial_x^\mu v\right\|_{r-\mu,\chi^{(\alpha,\beta)},A}.$$

For any real $\mu > 0$, the space $H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda)$ and its norm are defined by space interpolation. In particular, $H^r_{\chi^{(\alpha,\beta)},*}(\Lambda) = H^r_{\chi^{(\alpha,\beta)},*,1}(\Lambda)$ and $\|v\|_{r,\chi^{(\alpha,\beta)},*} = \|v\|_{r,\chi^{(\alpha,\beta)},*,1}$.

The first orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$ is a mapping such that for any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \quad \phi \in \mathcal{P}_N.$$

Lemma 3.9 (Theorem 2.3 of Guo [22]). For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and $r \geqslant 0$,

$$\|P_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.17}$$

In particular,

$$\|P_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \leqslant cN^{-1}|v|_{1,\chi^{(\alpha+1,\beta+1)}}. \tag{3.18}$$

As is well known, we usually consider the $H^r_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal projection in numerical analysis of differential equations. But in many practical problems, the coefficients of derivatives of different orders may degenerate in different ways. In these cases, it is not possible to compare the approximate solutions with the exact solutions in the Sobolev spaces. Whereas it might be carried out in certain Hilbert spaces. To do this, let $\alpha, \beta, \gamma, \delta > -1$, and introduce the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \leqslant \mu \leqslant 1$. For $\mu = 0$, $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \big\{v \mid v \text{ is measurable and } \|v\|_{1,\alpha,\beta,\gamma,\delta} < \infty\big\},$$

where

$$\|v\|_{1,\alpha,\beta,\gamma,\delta} = \big(|v|^2_{1,\chi^{(\alpha,\beta)}} + \|v\|^2_{\chi^{(\gamma,\delta)}}\big)^{1/2}.$$

For $0 < \mu < 1$, the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation. Its norm is denoted by $\|v\|_{\mu,\alpha,\beta,\gamma,\delta}$. Let

$$a_{\alpha,\beta,\gamma,\delta}(u, v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u, v)_{\chi^{(\gamma,\delta)}}, \quad \forall u, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

In particular, $a_{\alpha,\beta}(u, v) = a_{\alpha,\beta,\alpha,\beta}(u, v)$.

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N$ is a mapping such that for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}\big(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi\big) = 0, \quad \forall \phi \in \mathcal{P}_N.$$

In particular, $P^1_{N,\alpha,\beta} = P^1_{N,\alpha,\beta,\alpha,\beta}$.

Lemma 3.10 (Theorem 2.5 of Guo [22]). If (3.9) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\big\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\big\|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.19}$$

If, in addition,

$$\alpha \leqslant \gamma + 1, \qquad \beta \leqslant \delta + 1, \tag{3.20}$$

then for $0 \leqslant \mu \leqslant 1$,

$$\big\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\big\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.21}$$

Remark 3.3. As a special case of lemma 3.10, we get that for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| P^1_{N,\alpha,\beta} v - v \right\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.22}$$

In some practical problems arising in fluid dynamics, biology and other fields, the unknown functions vanish at one of the extreme points, say $x = -1$. So we need other orthogonal projections. Let

$${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \left\{ v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0 \right\}.$$

The orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta} : {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathcal{P}_N$ is a mapping such that for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}\left({}_0P^1_{N,\alpha,\beta,\gamma,\delta} v - v, \phi\right) = 0, \quad \forall \phi \in {}_0\mathcal{P}_N.$$

In particular, ${}_0P^1_{N,\alpha,\beta} = {}_0P^1_{N,\alpha,\beta,\alpha,\beta}$.

Lemma 3.11. If one of the following conditions holds,

$$\text{(i)} \quad \alpha \leqslant \gamma + 1, \quad \beta \leqslant \delta + 2, \quad 0 < \alpha < 1, \quad \beta < 1, \tag{3.23}$$

$$\text{(ii)} \quad \alpha \leqslant \gamma + 2, \quad \beta \leqslant 0, \qquad \delta \geqslant 0, \tag{3.24}$$

then for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\left\| {}_0P^1_{N,\alpha,\beta,\gamma,\delta} v - v \right\|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.25}$$

If, in addition, (3.20) holds, then for $0 \leqslant \mu \leqslant 1$,

$$\left\| {}_0P^1_{N,\alpha,\beta,\gamma,\delta} v - v \right\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.26}$$

We can prove this lemma by a similar argument as in the proof of theorem 2.6 of Guo [22]. Furthermore, we have the following result.

Lemma 3.12. If $\beta < 1$, then there exists a projection $\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N$ such that $\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} v(-1) = v(-1)$. Further, if the conditions for which (3.26) holds are fulfilled. Then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} v - v \right\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.27}$$

Moreover, if, in addition, $\gamma, \delta \leqslant 0$, then for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^d_{\chi^{(\alpha,\beta)},*}(\Lambda) \cap H^d(\Lambda)$ and $d > 1$,

$$\left\| \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} v \right\|_\infty \leqslant c\left(\|v\|_{d,\chi^{(\alpha,\beta)},*} + \|v\|_d + \|v\|_{1,\alpha,\beta,\gamma,\delta} \right). \tag{3.28}$$

Proof. Let Λ^* be any closed subset of Λ. Then for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, we have that $v \in C(\Lambda^*)$ and $|v(0)| \leqslant c\|v\|_{1,\alpha,\beta,\gamma,\delta}$. Thanks to $\beta < 1$, we have that for any $x_1, x_2 \in (-1, 0]$,

$$\begin{aligned}\big|v(x_1)\big| &\leqslant \big|v(x_2)\big| + \int_{x_1}^{x_2} \big|\partial_x v(x)\big|\,\mathrm{d}x\\ &\leqslant \big|v(x_2)\big| + c\left(\int_{-1}^{0} \big(\partial_x v(x)\big)^2 (1+x)^\beta\,\mathrm{d}x\right)^{1/2} \leqslant \big|v(x_2)\big| + \|v\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}$$

Letting $x_1 \to -1$ and $x_2 = 0$, we know that $v(x)$ is meaningful at $x = -1$, and $|v(-1)| \leqslant c\|v\|_{1,\alpha,\beta,\gamma,\delta}$. So we can define the desired projection as

$$\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} v(x) = v(-1) + {}_0P^1_{N,\alpha,\beta,\gamma,\delta}\big(v(x) - v(-1)\big). \tag{3.29}$$

Clearly, $\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} v(x) = v(x)$ for $x = -1$. Furthermore, it is not difficult to reach (3.27) by using (3.26) and (3.29).

We now prove (3.28). For any $u \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, we get from imbedding theorem that for $d > 1$,

$$\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u\big\|_\infty \leqslant \|u\|_{d/2} + c\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u - P_{N,0,0}u\big\|_{d/2} + \|P_{N,0,0}u - u\|_{d/2}.$$

By lemma 3.2 and lemma 3.11,

$$\begin{aligned}\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u - P_{N,0,0}u\big\|_{d/2} &\leqslant cN^d\big(\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u - u\big\| + \|P_{N,0,0}u - u\|\big)\\ &\leqslant cN^d\big(\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u - u\big\|_{\chi^{(\gamma,\delta)}} + \|P_{N,0,0}u - u\|\big)\\ &\leqslant c\big(\|u\|_{d,\chi^{(\alpha,\beta)},*} + \|u\|_{d,\chi^{(0,0)},*}\big) \leqslant c\big(\|u\|_{d,\chi^{(\alpha,\beta)},*} + \|u\|_d\big).\end{aligned}$$

Moreover, according to the property of the Legendre approximation,

$$\|P_{N,0,0}u - u\|_{d/2} \leqslant c\|u\|_{3d/4}.$$

So the above facts lead to that

$$\big\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} u\big\|_\infty \leqslant c\big(\|u\|_{d,\chi^{(\alpha,\beta)},*} + \|u\|_d\big).$$

Taking $u(x) = v(x) - v(-1)$ in the above inequality, we use (3.29) and the fact that $|v(-1)| \leqslant c\|v\|_{1,\alpha,\beta,\gamma,\delta}$ to obtain the desired result. □

In particular, for $\alpha = \gamma$ and $\beta = \delta$, we have the following approximation result. Let $\widehat{P}^1_{N,\alpha,\beta} = \widehat{P}^1_{N,\alpha,\beta,\alpha,\beta}$.

Lemma 3.13. If one of the following conditions holds,

$$\begin{aligned}&\text{(i)} \quad \alpha > -1, \quad \beta = 0,\\ &\text{(ii)} \quad -1 < \alpha, \beta < 1,\end{aligned} \tag{3.30}$$

then for any $v \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| {}_0P^1_{N,\alpha,\beta} v - v \right\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.31}$$

Moreover, for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| \widehat{P}^1_{N,\alpha,\beta} v - v \right\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.32}$$

Proof. If (i) of (3.30) holds, then (3.31) and (3.32) follows directly from lemma 3.11 and lemma 3.12 with $\alpha = \gamma$ and $\beta = \delta = 0$, respectively. We now prove (3.31) with condition (ii) of (3.30). We first consider the case $\mu = 1$. Let

$$\phi(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta} \partial_y v(y)\, \mathrm{d}y.$$

Clearly $\phi \in {}_0\mathcal{P}_N$. By the projection theorem, remark 3.1 and lemma 3.9,

$$\begin{aligned} &\left\| {}_0P^1_{N,\alpha,\beta} v - v \right\|_{1,\chi^{(\alpha,\beta)}} \\ &\quad \leqslant \|\phi - v\|_{1,\chi^{(\alpha,\beta)}} \leqslant c\|\partial_x\phi - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\quad \leqslant c\|P_{N-1,\alpha,\beta}\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\quad \leqslant cN^{1-r}\|\partial_x v\|_{r-1,\chi^{(\alpha,\beta)},A} \leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned}$$

We next deal with the case $\mu = 0$. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and consider an auxiliary problem. It is to find $w \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ such that

$$a_{\alpha,\beta}(w,z) = (g,z)_{\chi^{(\alpha,\beta)}}, \quad \forall z \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda). \tag{3.33}$$

Taking $z = w$ in (3.33), we get that

$$\|w\|_{1,\chi^{(\alpha,\beta)}} \leqslant c\|g\|_{\chi^{(\alpha,\beta)}}. \tag{3.34}$$

Now let $w(x)$ vary in $\mathcal{D}(\Lambda)$, and so in sense of distributions,

$$-\partial_x\big(\partial_x w(x)\chi^{(\alpha,\beta)}(x)\big) = \big(g(x) - w(x)\big)\chi^{(\alpha,\beta)}(x). \tag{3.35}$$

For simplicity of statements, let $\eta(x) = \partial_x w(x)\chi^{(\alpha,\beta)}(x)$. Then we know from (3.35) that for any $x_1, x_2 \in \Lambda$,

$$\big|\eta(x_2) - \eta(x_1)\big| \leqslant c\|g - w\|_{\chi^{(\alpha,\beta)}} \left(\int_{x_1}^{x_2} \chi^{(\alpha,\beta)}(x)\, \mathrm{d}x \right)^{1/2}.$$

Thus, $\eta(x_1) \to \eta(x_2)$ as $x_1 \to x_2$, i.e., $\eta(x) \in C(\overline{\Lambda})$. Furthermore, multiplying (3.35) by any $z \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ and integrating the resulting equality by parts, we obtain from (3.33) that

$$\begin{aligned} \eta(1)z(1) - \eta(-1)z(-1) = \int_\Lambda &\big(\partial_x w(x)\partial_x z(x) + w(x)z(x) \\ &- g(x)z(x)\big)\chi^{(\alpha,\beta)}(x)\, \mathrm{d}x = 0, \quad \forall z \in {}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda). \end{aligned}$$

Hence, $\eta(1) = 0$. Next, by (3.35),

$$-\partial_x^2 w(x) = \big((\alpha-\beta)+(\alpha+\beta)x\big)\chi^{(-1,-1)}(x)\partial_x w(x) + g(x) - w(x). \tag{3.36}$$

Let Λ_1 and Λ_2 be the same as before. It can be verified that

$$\big\|\partial_x^2 w\big(1-x^2\big)^{1/2}\big\|^2_{\chi^{(\alpha,\beta)}} \leqslant D_1 + D_2,$$

where $D_1 = D_1(\Lambda_1) + D_1(\Lambda_2)$, and

$$D_1(\Lambda_j) = 8\big(\alpha^2+\beta^2\big)\int_{\Lambda_j}\big(\partial_x w(x)\big)^2\chi^{(\alpha-1,\beta-1)}(x)\,\mathrm{d}x, \quad j=1,2,$$
$$D_2 = 2\int_{\Lambda}\big(g(x)-w(x)\big)^2\chi^{(\alpha+1,\beta+1)}(x)\,\mathrm{d}x.$$

Clearly, $D_2 \leqslant c\|g-w\|^2_{\chi^{(\alpha,\beta)}}$. So it remains to estimate D_1. Thanks to $\eta(1) = 0$ and $|\alpha| < 1$, we get from (3.11), (3.34) and (3.35) that

$$\begin{aligned}
D_1(\Lambda_1) &= 8\big(\alpha^2+\beta^2\big)\int_{\Lambda_1}\chi^{(-\alpha-1,-\beta-1)}(x)\bigg(\int_x^1\big(g(y)-w(y)\big)\chi^{(\alpha,\beta)}(y)\,\mathrm{d}y\bigg)^2\mathrm{d}x \\
&\leqslant c\int_{\Lambda_1}\bigg(\frac{1}{1-x}\int_x^1\big(g(y)-w(y)\big)\chi^{(\alpha,\beta)}(y)\,\mathrm{d}y\bigg)^2(1-x)^{-\alpha}\,\mathrm{d}x \\
&\leqslant c\int_{\Lambda_1}\big(g(x)-w(x)\big)^2\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \leqslant c\|g\|^2_{\chi^{(\alpha,\beta)}}.
\end{aligned} \tag{3.37}$$

If $\beta > 0$, then $\eta(-1) = 0$. In this case, a similar argument as in the derivation of (3.37) leads to that

$$D_1(\Lambda_2) \leqslant c\|g\|^2_{\chi^{(\alpha,\beta)}}.$$

If $-1 < \beta < 0$, then we deduce from (3.11), (3.34) and (3.35) that

$$\begin{aligned}
&D_1(\Lambda_2) \\
&\quad = 8\big(\alpha^2+\beta^2\big)\int_{\Lambda_2}\chi^{(-\alpha-1,-\beta-1)}(x)\bigg(\int_{-1}^x\big(g(y)-w(y)\big)\chi^{(\alpha,\beta)}(y)\,\mathrm{d}y - \eta(-1)\bigg)^2\mathrm{d}x \\
&\quad \leqslant c\bigg(\int_{\Lambda_2}(1+x)^{-\beta-1}\bigg(\int_{-1}^x\big(g(y)-w(y)\big)\chi^{(\alpha,\beta)}(y)\,\mathrm{d}y\bigg)^2\mathrm{d}x + \eta^2(-1)\bigg) \\
&\quad \leqslant c\bigg(\int_{\Lambda_2}\bigg(\frac{1}{1+x}\int_{-1}^x\big(g(y)-w(y)\big)\chi^{(\alpha,\beta)}(y)\,\mathrm{d}y\bigg)^2(1+x)^{-\beta}\,\mathrm{d}x + \eta^2(-1)\bigg) \\
&\quad \leqslant c\bigg(\int_{\Lambda_2}\big(g(x)-w(x)\big)^2\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x + \eta^2(-1)\bigg) \\
&\quad \leqslant c\big(\|g\|^2_{\chi^{(\alpha,\beta)}} + \eta^2(-1)\big).
\end{aligned}$$

Besides, due to $\eta(1) = 0$, we have from (3.34) and (3.35) that

$$\eta^2(-1) \leqslant \left(\int_\Lambda \left|\partial_x \eta(x)\right| \mathrm{d}x \right)^2 \leqslant \int_\Lambda \left(\partial_x \eta(x)\right)^2 \chi^{(-\alpha,-\beta)}(x)\,\mathrm{d}x \int_\Lambda \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x$$
$$\leqslant c \int_\Lambda \left(g(x) - w(x)\right)^2 \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \leqslant c\|g\|^2_{\chi^{(\alpha,\beta)}}.$$

A combination of the above inequalities leads to that

$$\left\|\partial_x^2 w(1-x)^{1/2}\right\|_{\chi^{(\alpha,\beta)}} \leqslant c\|g\|_{\chi^{(\alpha,\beta)}}.$$

Therefore, by (3.31) with $\mu = 1$,

$$\left\|{}_0P^1_{N,\alpha,\beta} w - w\right\|_{1,\chi^{(\alpha,\beta)}} \leqslant cN^{-1}\|w\|_{2,\chi^{(\alpha,\beta)},*} \leqslant cN^{-1}\|\partial_x w\|_{1,\chi^{(\alpha,\beta)},A}$$
$$\leqslant cN^{-1}\|g\|_{\chi^{(\alpha,\beta)}}.$$

Taking $z = {}_0P^1_{N,\alpha,\beta} v - v$ in (3.33), we obtain that

$$\begin{aligned}\left|\left({}_0P^1_{N,\alpha,\beta} v - v, g\right)_{\chi^{(\alpha,\beta)}}\right| &= \left|a_{\alpha,\beta}\left({}_0P^1_{N,\alpha,\beta} v - v, w\right)\right| \\ &= \left|a_{\alpha,\beta}\left({}_0P^1_{N,\alpha,\beta} v - v, {}_0P^1_{N,\alpha,\beta} w - w\right)\right| \\ &\leqslant \left\|{}_0P^1_{N,\alpha,\beta} v - v\right\|_{1,\chi^{(\alpha,\beta)}} \left\|{}_0P^1_{N,\alpha,\beta} w - w\right\|_{1,\chi^{(\alpha,\beta)}} \\ &\leqslant cN^{-r}\|g\|_{\chi^{(\alpha,\beta)}}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.38}$$

Consequently,

$$\begin{aligned}\left\|{}_0P^1_{N,\alpha,\beta} v - v\right\|_{\chi^{(\alpha,\beta)}} &= \sup_{g\in L^2_{\chi^{(\alpha,\beta)}}, g\neq 0} \frac{|({}_0P^1_{N,\alpha,\beta} v - v, g)_{\chi^{(\alpha,\beta)}}|}{\|g\|_{\chi^{(\alpha,\beta)}}} \\ &\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.39}$$

Moreover, (3.31) with $\alpha < 1$ and $\beta = 0$ is a special case of lemma 3.11 with $\alpha = \gamma$ and $\beta = \delta = 0$. Therefore, (3.31) follows from the previous statements and space interpolation. We now prove (3.32) with condition (ii) of (3.30). By (3.31),

$$\left\|\widehat{P}^1_{N,\alpha,\beta} v - v\right\|_{\mu,\chi^{(\alpha,\beta)}} = \left\|{}_0P^1_{N,\alpha,\beta}\left(v(x) - v(-1)\right) - \left(v(x) - v(-1)\right)\right\|_{\mu,\chi^{(\alpha,\beta)}}$$
$$\leqslant cN^{\mu-r}\left\|v(x) - v(-1)\right\|_{r,\chi^{(\alpha,\beta)},*} \leqslant cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

This implies the desired result. □

When we study movements of fluid flows in bounded domains with fixed non-slip walls, the populations of budworms in bounded forests with lethal boundary conditions, and some other problems, we encounter homogenous boundary conditions. In these cases, we have to consider other projections. To do this, let

$$\widetilde{a}_{\alpha,\beta}(u, v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}}, \qquad \overline{a}_{\alpha,\beta}(u, v) = \left(\partial_x u, \partial_x\left(v\chi^{(\alpha,\beta)}(x)\right)\right).$$

The orthogonal projection $\widetilde{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\widetilde{a}_{\alpha,\beta}\big(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, \phi\big) = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

The orthogonal projection $\overline{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\overline{a}_{\alpha,\beta}\big(\overline{P}^{1,0}_{N,\alpha,\beta}v - v, \phi\big) = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

We first analyze some properties of the bilinear form $\overline{a}_{\alpha,\beta}(\cdot,\cdot)$.

Lemma 3.14. If $-1 < \alpha, \beta \leqslant 0$ or $0 \leqslant \alpha, \beta < 1$, then the bilinear form $\overline{a}_{\alpha,\beta}(u, v)$ is continuous on $H^1_{\chi^{(\alpha,\beta)}}(\Lambda) \times H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ and elliptic on $H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \times H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$.

Proof. By the Cauchy–Schwartz inequality and lemma 3.7,

$$\begin{aligned}\overline{a}_{\alpha,\beta}(u, v) &= \int_\Lambda \partial_x u(x)\partial_x v(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x + \int_\Lambda \partial_x u(x)v(x)\partial_x\big(\chi^{(\alpha,\beta)}(x)\big)\,\mathrm{d}x \\ &\leqslant \|\partial_x u\|_{\chi^{(\alpha,\beta)}}\|\partial_x v\|_{\chi^{(\alpha,\beta)}} + 4\big(\alpha^2+\beta^2\big)\|\partial_x u\|_{\chi^{(\alpha,\beta)}}\|v\|_{\chi^{(\alpha-2,\beta-2)}} \\ &\leqslant c\|\partial_x u\|_{\chi^{(\alpha,\beta)}}\|\partial_x v\|_{\chi^{(\alpha,\beta)}}.\end{aligned}$$

On the other hand, integrating by parts yields that

$$\begin{aligned}\overline{a}_{\alpha,\beta}(u, u) &= \|\partial_x u\|^2_{\chi^{(\alpha,\beta)}} + \frac{1}{2}\int_\Lambda \partial_x\big(u^2(x)\big)\partial_x\big(\chi^{(\alpha,\beta)}(x)\big)\,\mathrm{d}x \\ &= \|\partial_x u\|^2_{\chi^{(\alpha,\beta)}} - \frac{1}{2}\int_\Lambda u^2(x)\partial_x^2\big(\chi^{(\alpha,\beta)}(x)\big)\,\mathrm{d}x \\ &= \|\partial_x u\|^2_{\chi^{(\alpha,\beta)}} + \frac{1}{2}\int_\Lambda u^2(x)W(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x,\end{aligned}$$

where

$$W(x) = -\partial_x^2\big(\chi^{(\alpha,\beta)}(x)\big)\chi^{(-\alpha,-\beta)}(x).$$

We now determine the ranges of α and β such that $W(x) \geqslant 0$ for all $x \in \overline{\Lambda}$. Let $f(x) = (1-x^2)^2 W(x)$. A calculation shows that

$$f(x) = -(\alpha+\beta)(\alpha+\beta-1)x^2 + 2(\beta-\alpha)(\alpha+\beta-1)x + \alpha + \beta - (\beta-\alpha)^2.$$

By the properties of the quadratic function, we find that $f(x) \geqslant 0$ for all $x \in \overline{\Lambda}$, if

$$\begin{cases}(\alpha+\beta)(\alpha+\beta-1) \geqslant 0, \\ f(-1) = -4\beta^2 + 4\beta \geqslant 0, \\ f(1) = -4\alpha^2 + 4\alpha \geqslant 0,\end{cases} \tag{3.40}$$

or

$$\begin{cases}(\alpha+\beta)(\alpha+\beta-1)\leqslant 0,\\ 4(\beta-\alpha)^2(\alpha+\beta-1)^2+4(\alpha+\beta)(\alpha+\beta-1)\big(\alpha+\beta-(\beta-\alpha)^2\big)\leqslant 0.\end{cases}\tag{3.41}$$

Solving (3.40) and (3.41) yields that $0\leqslant\alpha,\beta\leqslant 1$. Therefore, if $0\leqslant\alpha,\beta\leqslant 1$, then we obtain that

$$\overline{a}_{\alpha,\beta}(u,u)\geqslant c\|u\|^2_{1,\chi^{(\alpha,\beta)}}.\tag{3.42}$$

Furthermore, if $-1<\alpha,\beta\leqslant 0$, then we set $w=u\chi^{(\alpha,\beta)}$. So we know from lemma 3.8 that $w\in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$. Hence, by (3.42),

$$\overline{a}_{\alpha,\beta}(u,u)=\overline{a}_{-\alpha,-\beta}(w,w)\geqslant c\|w\|^2_{1,\chi^{(-\alpha,-\beta)}}\geqslant c\|u\|^2_{1,\chi^{(\alpha,\beta)}}.$$

Finally, a combination of (3.42) and the above inequality leads to the conclusion. □

To estimate the difference between $\widetilde{P}^{1,0}_{N,\alpha,\beta}v$ and v, we need a useful lemma. For $-1<\alpha,\beta<1$, let

$$\mathcal{U}_{N,\alpha,\beta}(\Lambda)=\big\{v\mid v=\chi^{(\alpha,\beta)}\phi,\ \phi\in\mathcal{P}_{N-1}\big\}.$$

The orthogonal projection $\mathcal{T}_{N,\alpha,\beta}:L^2_{\chi^{(-\alpha,-\beta)}}(\Lambda)\to\mathcal{U}_{N,\alpha,\beta}(\Lambda)$ is a mapping such that for any $v\in L^2_{\chi^{(-\alpha,-\beta)}}(\Lambda)$,

$$(\mathcal{T}_{N,\alpha,\beta}v-v,\phi)_{\chi^{(-\alpha,-\beta)}}=0,\quad\forall\phi\in\mathcal{U}_{N,\alpha,\beta}(\Lambda).$$

We shall compare $\mathcal{T}_{N,\alpha,\beta}v$ with v in the norm $\|\cdot\|_{\chi^{(-\alpha,-\beta)}}$.

Lemma 3.15. If $-1<\alpha,\beta<1$, then for any $v\in H^1_{\chi^{(-\alpha,-\beta)}}(\Lambda)$,

$$\|\mathcal{T}_{N,\alpha,\beta}v-v\|_{\chi^{(-\alpha,-\beta)}}\leqslant cN^{\sigma(\alpha,\beta)-1}\|v\|_{1,\chi^{(-\alpha,-\beta)}},\tag{3.43}$$

where $\sigma(\alpha,\beta)=\max(\alpha,\beta,0)$.

Proof. Lemma 3.5 implies that $H^1_{\chi^{(-\alpha,-\beta)}}(\Lambda)\subset C(\overline{\Lambda})$. So we can define an affine function as

$$v^*(x)=\frac{1}{2}v(1)(1+x)+\frac{1}{2}v(-1)(1-x),\quad x\in\overline{\Lambda}.\tag{3.44}$$

Let $v^0(x)=v(x)-v^*(x)$. Obviously $v^0\in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$. By remark 3.2,

$$\begin{aligned}\big\|v^0\big\|_{1,\chi^{(-\alpha,-\beta)}}&\leqslant c\big|v^0\big|_{1,\chi^{(-\alpha,-\beta)}}\leqslant c\big(\|\partial_x v\|_{\chi^{(-\alpha,-\beta)}}+\big|v(1)-v(-1)\big|\big)\\ &\leqslant c\left(\|\partial_x v\|_{\chi^{(-\alpha,-\beta)}}+\int_\Lambda|\partial_x v|\,\mathrm{d}x\right)\leqslant c\|\partial_x v\|_{\chi^{(-\alpha,-\beta)}}.\end{aligned}\tag{3.45}$$

Furthermore, let

$$v^0_N(x)=\chi^{(\alpha,\beta)}(x)P_{N-1,\alpha,\beta}\big(v^0(x)\chi^{(-\alpha,-\beta)}(x)\big).$$

Then $v_N^0 \in \mathcal{U}_{N,\alpha,\beta}(\Lambda)$. By (3.18),

$$\begin{aligned}\left\|v_N^0 - v^0\right\|_{\chi^{(-\alpha,-\beta)}} &= \left\|P_{N-1,\alpha,\beta}\big(v^0\chi^{(-\alpha,-\beta)}\big) - v^0\chi^{(-\alpha,-\beta)}\right\|_{\chi^{(\alpha,\beta)}} \\ &\leqslant cN^{-1}\left\|\partial_x\big(v^0\chi^{(-\alpha,-\beta)}\big)\right\|_{\chi^{(\alpha+1,\beta+1)}} \\ &\leqslant cN^{-1}\big(\left\|\partial_x v^0\right\|_{\chi^{(-\alpha+1,-\beta+1)}} + \left\|v^0\right\|_{\chi^{(-\alpha-1,-\beta-1)}}\big).\end{aligned}$$

Clearly, by (3.45),

$$\left\|\partial_x v^0\right\|_{\chi^{(-\alpha+1,-\beta+1)}} \leqslant c\|v\|_{1,\chi^{(-\alpha,-\beta)}}.$$

By virtue of lemma 3.7 and (3.45),

$$\left\|v^0\right\|_{\chi^{(-\alpha-1,-\beta-1)}} \leqslant c\left\|v^0\right\|_{\chi^{(-\alpha-2,-\beta-2)}} \leqslant c\left\|v^0\right\|_{1,\chi^{(-\alpha,-\beta)}} \leqslant c\|v\|_{1,\chi^{(-\alpha,-\beta)}}.$$

Therefore,

$$\left\|v_N^0 - v^0\right\|_{\chi^{(-\alpha,-\beta)}} \leqslant cN^{-1}\|v\|_{1,\chi^{(-\alpha,-\beta)}}. \tag{3.46}$$

Next, we consider the bound of $\|\mathcal{I}_{N,\alpha,\beta}v^* - v^*\|_{\chi^{(-\alpha,-\beta)}}$. It suffices to estimate the bounds $\|\mathcal{I}_{N,\alpha,\beta}w - w\|_{\chi^{(-\alpha,-\beta)}}$, $w = 1, x$. Due to the definition of $\mathcal{I}_{N,\alpha,\beta}$,

$$\mathcal{I}_{N,\alpha,\beta}1 - 1 = -\sum_{l=N}^{\infty} d_l\chi^{(\alpha,\beta)}(x)J_l^{(\alpha,\beta)}(x)$$

with

$$d_l = \left\|\chi^{(\alpha,\beta)}J_l^{(\alpha,\beta)}\right\|_{\chi^{(-\alpha,-\beta)}}^{-2}\int_\Lambda J_l^{(\alpha,\beta)}(x)\,\mathrm{d}x = \frac{1}{\gamma_l^{(\alpha,\beta)}}\int_\Lambda J_l^{(\alpha,\beta)}(x)\,\mathrm{d}x.$$

According to (2.7),

$$\begin{aligned}\int_\Lambda J_l^{(\alpha,\beta)}(x)\,\mathrm{d}x = {}& a_l\big(J_{l+1}^{(\alpha,\beta)}(1) - J_{l+1}^{(\alpha,\beta)}(-1)\big) + b_l\big(J_l^{(\alpha,\beta)}(1) - J_l^{(\alpha,\beta)}(-1)\big) \\ &+ c_l\big(J_{l-1}^{(\alpha,\beta)}(1) - J_{l-1}^{(\alpha,\beta)}(-1)\big).\end{aligned}$$

By (2.3) and (3.5),

$$\gamma_l^{(\alpha,\beta)} \leqslant cl^{-1}, \qquad b_l \leqslant cl^{-2}, \qquad \left|J_l^{(\alpha,\beta)}(1)\right| \leqslant cl^{\alpha}, \qquad \left|J_l^{(\alpha,\beta)}(-1)\right| \leqslant cl^{\beta}.$$

Moreover, by using (2.3) and the expressions of a_l and c_l in (2.7), we derive from a direct calculation that

$$a_l J_{l+1}^{(\alpha,\beta)}(1) + c_l J_{l-1}^{(\alpha,\beta)}(1) = \frac{2(l+\alpha)\Gamma(l+\alpha)p(l)}{(l+1)!(l+\alpha+\beta)\Gamma(\alpha+1)\prod_{k=0}^{2}(2l+\alpha+\beta+k)},$$

where $p(l)$ is a polynomial of degree 3, i.e.,

$$p(l) = (l+\alpha+1)(l+\alpha+\beta)(l+\alpha+\beta+1)(2l+\alpha+\beta) - l(l+1)(l+\beta)(2l+\alpha+\beta+2).$$

Thus by (3.5),

$$\left|a_l J_{l+1}^{(\alpha,\beta)}(1) + c_l J_{l-1}^{(\alpha,\beta)}(1)\right| \leqslant c l^{\alpha-2}.$$

Similarly,

$$\left|a_l J_{l+1}^{(\alpha,\beta)}(-1) + c_l J_{l-1}^{(\alpha,\beta)}(-1)\right| \leqslant c l^{\beta-2}.$$

Let $\eta(\alpha,\beta) = \max(\alpha,\beta)$. We deduce from the above statements that $|d_l| \leqslant c l^{\eta(\alpha,\beta)-1}$. Thus

$$\|\mathcal{I}_{N,\alpha,\beta}1 - 1\|_{\chi^{(-\alpha,-\beta)}} \leqslant c\left(\sum_{l=N}^{\infty} l^{2\eta(\alpha,\beta)-3}\right)^{1/2} \leqslant c N^{\eta(\alpha,\beta)-1}. \tag{3.47}$$

Similarly,

$$\|\mathcal{I}_{N,\alpha,\beta}x - x\|_{\chi^{(-\alpha,-\beta)}} \leqslant c N^{\eta(\alpha,\beta)-1}. \tag{3.48}$$

Finally, we derive from the projection theorem, lemma 3.5 and (3.46)–(3.48) that

$$\begin{aligned}
\|\mathcal{I}_{N,\alpha,\beta}v - v\|_{\chi^{(-\alpha,-\beta)}} &\leqslant c\big(\big\|\mathcal{I}_{N,\alpha,\beta}v^0 - v^0\big\|_{\chi^{(-\alpha,-\beta)}} + \big\|\mathcal{I}_{N,\alpha,\beta}v^* - v^*\big\|_{\chi^{(-\alpha,-\beta)}}\big) \\
&\leqslant c\big(\big\|v_N^0 - v^0\big\|_{\chi^{(-\alpha,-\beta)}} + c\big(\big|v(1)\big| + \big|v(-1)\big|\big) \\
&\qquad \times \big(\|\mathcal{I}_{N,\alpha,\beta}1 - 1\|_{\chi^{(-\alpha,-\beta)}} + \|\mathcal{I}_{N,\alpha,\beta}x - x\|_{\chi^{(-\alpha,-\beta)}}\big)\big) \\
&\leqslant c N^{\sigma(\alpha,\beta)-1}\|v\|_{1,\chi^{(-\alpha,-\beta)}}
\end{aligned}$$

which completes the proof. □

Lemma 3.16. If $-1 < \alpha, \beta < 1$, then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\big\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\big\|_{1,\chi^{(\alpha,\beta)}} \leqslant c N^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.49}$$

If, in addition, $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, then for $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\big\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\big\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant c N^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.50}$$

Proof. We first prove (3.49). Let

$$v^*(x) = \int_{-1}^{x}\left(P_{N-1,\alpha,\beta}\partial_y v(y) - \frac{1}{2}\int_{-1}^{1} P_{N-1,\alpha,\beta}\partial_z v(z)\,\mathrm{d}z\right)\mathrm{d}y.$$

Clearly $v^* \in \mathcal{P}_N^0$. By the projection theorem, remark 3.2 and lemma 3.9,

$$\begin{aligned}
&\big\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\big\|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant c\big\|v - v^*\big\|_{1,\chi^{(\alpha,\beta)}} \leqslant c\big\|\partial_x v - \partial_x v^*\big\|_{\chi^{(\alpha,\beta)}} \\
&\quad \leqslant c\left(\|P_{N-1,\alpha,\beta}\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} + \left|\int_{-1}^{1}\big(P_{N-1,\alpha,\beta}\partial_y v(y) - \partial_y v(y)\big)\mathrm{d}y\right|\right) \\
&\quad \leqslant c\|P_{N-1,\alpha,\beta}\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \leqslant c N^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}$$

The above estimate completes the proof of (3.49).

We now turn to prove (3.50) with $-1 < \alpha, \beta \leqslant 0$ and $\mu = 0$. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and consider an auxiliary problem. It is to find $w \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ such that

$$\widetilde{a}_{\alpha,\beta}(w,z) = (g,z)_{\chi^{(\alpha,\beta)}}, \quad \forall z \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda). \tag{3.51}$$

In sense of distributions,

$$-\partial_x\big(\chi^{(\alpha,\beta)}(x)\partial_x w(x)\big) = g(x)\chi^{(\alpha,\beta)}(x). \tag{3.52}$$

Let $u(x) = \chi^{(\alpha,\beta)}(x)\partial_x w(x)$. We take $w = z$ in (3.51) and integrate the resulting equation by parts. Due to $\|u\|_{\chi^{(-\alpha,-\beta)}} = |w|_{1,\chi^{(\alpha,\beta)}}$ and the Poincaré inequality, we assert that $\|u\|_{\chi^{(-\alpha,-\beta)}} \leqslant c\|g\|_{\chi^{(\alpha,\beta)}}$. Moreover, by taking the $L^2_{\chi^{(-\alpha,-\beta)}}$-norms of both sides of (3.52), we obtain that

$$\|\partial_x u\|_{\chi^{(-\alpha,-\beta)}} = \big\|-g(x)\chi^{(\alpha,\beta)}(x)\big\|_{\chi^{(-\alpha,-\beta)}} \leqslant c\|g\|_{\chi^{(\alpha,\beta)}}.$$

Next, let $\mathcal{T}_{N,\alpha,\beta}$ be the same as in lemma 3.15, and

$$\begin{aligned} u_N(x) &= \mathcal{T}_{N,\alpha,\beta}u(x) - \frac{1}{2}\chi^{(\alpha,\beta)}(x)\int_\Lambda \mathcal{T}_{N,\alpha,\beta}u(x)\chi^{(-\alpha,-\beta)}(x)\,\mathrm{d}x, \\ w_N(x) &= \int_{-1}^x u_N(y)\chi^{(-\alpha,-\beta)}(y)\,\mathrm{d}y. \end{aligned}$$

Clearly, $u_N \in \mathcal{U}_{N,\alpha,\beta}(\Lambda)$ and $w_N \in \mathcal{P}^0_N$. It is noted that

$$\int_\Lambda \mathcal{T}_{N,\alpha,\beta}u(x)\chi^{(-\alpha,-\beta)}(x)\,\mathrm{d}x = \int_\Lambda \big(\mathcal{T}_{N,\alpha,\beta}u(x) - u(x)\big)\chi^{(-\alpha,-\beta)}(x)\,\mathrm{d}x.$$

For simplicity, we denote the right side of the above formula by $\eta(x)$. Then by lemma 3.15,

$$\begin{aligned} \|\partial_x w - \partial_x w_N\|_{\chi^{(\alpha,\beta)}} &= \big\|\chi^{(-\alpha,-\beta)}u - \chi^{(-\alpha,-\beta)}u_N\big\|_{\chi^{(\alpha,\beta)}} \\ &= \|u - u_N\|_{\chi^{(-\alpha,-\beta)}} \\ &= \big\|u - \mathcal{T}_{N,\alpha,\beta}u + \tfrac{1}{2}\chi^{(\alpha,\beta)}\eta\big\|_{\chi^{(-\alpha,-\beta)}} \\ &\leqslant \|\mathcal{T}_{N,\alpha,\beta}u - u\|_{\chi^{(-\alpha,-\beta)}} + c|\eta(x)| \\ &\leqslant c\|\mathcal{T}_{N,\alpha,\beta}u - u\|_{\chi^{(-\alpha,-\beta)}} \\ &\leqslant cN^{-1}\|u\|_{1,\chi^{(-\alpha,-\beta)}} \leqslant cN^{-1}\|g\|_{\chi^{(\alpha,\beta)}}. \end{aligned} \tag{3.53}$$

Taking $z = \widetilde{P}^{1,0}_{N,\alpha,\beta}v - v$ in (3.51), we obtain from (3.49) and (3.53) that

$$\begin{aligned} \big|\big(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, g\big)_{\chi^{(\alpha,\beta)}}\big| &= \big|\widetilde{a}_{\alpha,\beta}\big(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, w\big)\big| \\ &= \big|\widetilde{a}_{\alpha,\beta}\big(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, w - w_N\big)\big| \\ &\leqslant \big\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\big\|_{1,\chi^{(\alpha,\beta)}}\|\partial_x w - \partial_x w_N\|_{\chi^{(\alpha,\beta)}} \\ &\leqslant cN^{-r}\|g\|_{\chi^{(\alpha,\beta)}}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.54}$$

Consequently,

$$\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} = \sup_{g\in L^2_{\chi^{(\alpha,\beta)}}, g\neq 0} \frac{|(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, g)_{\chi^{(\alpha,\beta)}}|}{\|g\|_{\chi^{(\alpha,\beta)}}}$$
$$\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.55}$$

Finally, the result (3.50) for $0 \leqslant \mu \leqslant 1$ follows from (3.49), (3.55) and space interpolation.

We now consider the case with $0 < \alpha, \beta < 1$ and $\mu = 0$. Taking $z = w$ in (3.51), we get from remark 3.2 that $\|w\|_{1,\chi^{(\alpha,\beta)}} \leqslant c\|g\|_{\chi^{(\alpha,\beta)}}$. Also, in sense of distributions, (3.52) holds, and $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$ as $|x| \to 1$. By a similar argument as in the last part of the proof of lemma 3.13, we assert that (3.55) is also valid for $0 < \alpha, \beta < 1$. Then we reach (3.50) by space interpolation. □

Remark 3.4. The result (3.18) and the derivation of (3.49) imply that

$$\|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v\|_{1,\chi^{(\alpha,\beta)}} \leqslant cN^{-1}\|\partial_x^2 v\|_{\chi^{(\alpha+1,\beta+1)}}. \tag{3.56}$$

Now, we estimate the difference between $\overline{P}^{1,0}_{N,\alpha,\beta}v$ and v.

Lemma 3.17. If $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\|\overline{P}^{1,0}_{N,\alpha,\beta}v - v\|_{1,\chi^{(\alpha,\beta)}} \leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.57}$$

Proof. For any $\phi \in \mathcal{P}^0_N$, we get from lemma 3.14 that

$$c\|\overline{P}^{1,0}_{N,\alpha,\beta}v - v\|^2_{1,\chi^{(\alpha,\beta)}} \leqslant \bar{a}_{\alpha,\beta}\big(\overline{P}^{1,0}_{N,\alpha,\beta}v - v, \overline{P}^{1,0}_{N,\alpha,\beta}v - v\big) = \bar{a}_{\alpha,\beta}\big(\overline{P}^{1,0}_{N,\alpha,\beta}v - v, \phi - v\big)$$
$$\leqslant c\|\overline{P}^{1,0}_{N,\alpha,\beta}v - v\|_{1,\chi^{(\alpha,\beta)}}\|\phi - v\|_{1,\chi^{(\alpha,\beta)}}.$$

Thus

$$\|\overline{P}^{1,0}_{N,\alpha,\beta}v - v\|_{1,\chi^{(\alpha,\beta)}} \leqslant c\inf_{\phi\in\mathcal{P}^0_N}\|\phi - v\|_{1,\chi^{(\alpha,\beta)}}.$$

Taking $\phi = \widetilde{P}^{1,0}_{N,\alpha,\beta}v$ in the above inequality, we get (3.57) by using lemma 3.16. □

The following result plays an important role in analysis of Jacobi interpolation approximation.

Lemma 3.18. If $-1 < \alpha, \beta < 1$, then there exists a projection $\widetilde{P}^1_{N,\alpha,\beta} : H^1_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$, such that

$$\widetilde{P}^1_{N,\alpha,\beta}v(1) = v(1), \qquad \widetilde{P}^1_{N,\alpha,\beta}v(-1) = v(-1), \tag{3.58}$$

and for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\big\| \widetilde{P}^1_{N,\alpha,\beta} v - v \big\|_{1,\chi^{(\alpha,\beta)}} \leqslant cN^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.59}$$

If, in addition, $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\big\| \widetilde{P}^1_{N,\alpha,\beta} v - v \big\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.60}$$

Proof. By lemma 3.5, we can define an affine function $v^*(x)$ as in (3.44). Clearly, $v - v^* \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$. Now, let

$$\widetilde{P}^1_{N,\alpha,\beta} v(x) = v^*(x) + \widetilde{P}^{1,0}_{N,\alpha,\beta}\big(v(x) - v^*(x)\big). \tag{3.61}$$

Obviously, (3.58) holds. By remark 3.2,

$$\begin{aligned} \|v - v^*\|_{1,\chi^{(\alpha,\beta)}} &\leqslant c|v - v^*|_{1,\chi^{(\alpha,\beta)}} \leqslant c\big(\|\partial_x v\|_{\chi^{(\alpha,\beta)}} + |v(1) - v(-1)|\big) \\ &\leqslant c\left(\|\partial_x v\|_{\chi^{(\alpha,\beta)}} + \int_{-1}^{1} |\partial_x v| \,\mathrm{d}x\right) \leqslant c\|v\|_{1,\chi^{(\alpha,\beta)},*}. \end{aligned}$$

In view of this, we get from (3.49) and (3.61) that

$$\begin{aligned} \big\| \widetilde{P}^1_{N,\alpha,\beta} v - v \big\|_{1,\chi^{(\alpha,\beta)}} &= \big\| \widetilde{P}^{1,0}_{N,\alpha,\beta}(v - v^*) - (v - v^*) \big\|_{1,\chi^{(\alpha,\beta)}} \\ &\leqslant c\big\| \partial_x (v - v^*) \big\|_{\chi^{(\alpha,\beta)}} \leqslant c\|v\|_{1,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.62}$$

This implies (3.59) for $r = 1$. If $r \geqslant 2$, then we have from (3.49) and (3.61) that

$$\begin{aligned} \big\| \widetilde{P}^1_{N,\alpha,\beta} v - v \big\|_{1,\chi^{(\alpha,\beta)}} &= \big\| \widetilde{P}^{1,0}_{N,\alpha,\beta}(v - v^*) - (v - v^*) \big\|_{1,\chi^{(\alpha,\beta)}} \\ &\leqslant cN^{1-r} \|v - v^*\|_{r,\chi^{(\alpha,\beta)},*} \leqslant cN^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned}$$

Therefore, we can get (3.59) by using the above fact, (3.62) and space interpolation. Furthermore, (3.60) follows from (3.50), (3.61) and a similar argument as in the above lines. □

There exists a more precise result, stated below.

Lemma 3.19. If $0 < \alpha, \beta < 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$\big\| \widetilde{P}^1_{N,\alpha,\beta} v - v \big\|_{\chi^{(\alpha-1,\beta-1)}} \leqslant cN^{-r} \|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.63}$$

Proof. Let $g \in L^2_{\chi^{(\alpha-1,\beta-1)}}(\Lambda)$ and consider the auxiliary problem

$$\widetilde{a}_{\alpha,\beta}(w, z) = (g, z)_{\chi^{(\alpha-1,\beta-1)}}, \quad \forall z \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda). \tag{3.64}$$

Clearly, $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$, as $|x| \to 1$. A similar process as in the proof of the last part of lemma 3.13 yields that

$$\|w\|_{2,\chi^{(\alpha,\beta)},*} \leqslant c\|g\|_{\chi^{(\alpha-1,\beta-1)}}. \tag{3.65}$$

Let $v^*(x)$ be the same as in (3.44). The definition (3.61) implies that

$$\widetilde{a}_{\alpha,\beta}\big(\widetilde{P}^1_{N,\alpha,\beta}v-v,\phi\big)=\widetilde{a}_{\alpha,\beta}\big(\widetilde{P}^{1,0}_{N,\alpha,\beta}(v-v^*)-(v-v^*),\phi\big)=0,\quad \forall\phi\in\mathcal{P}^0_N(\Lambda). \tag{3.66}$$

By taking $z=\widetilde{P}^1_{N,\alpha,\beta}v-v$ in (3.64) and using (3.62), (3.65) and (3.66), we derive that

$$\big|\big(\widetilde{P}^1_{N,\alpha,\beta}v-v,g\big)_{\chi^{(\alpha-1,\beta-1)}}\big|\leqslant cN^{-r}\|g\|_{\chi^{(\alpha-1,\beta-1)}}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Then the conclusion follows from a duality argument. □

4. Jacobi interpolation approximations

In this section, we establish the main results of this paper. We shall compare $\mathcal{I}_{Z,N,\alpha,\beta}v$ with v in various norms. We first consider the Jacobi–Gauss interpolation, and begin with a result related to the stability of interpolation.

Theorem 4.1. For any $v\in H^1_{\chi^{(\alpha+1,\beta+1)}}(\Lambda)\cap L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\|\mathcal{I}_{G,N,\alpha,\beta}v\|_{\chi^{(\alpha,\beta)}}\leqslant c\big(\|v\|_{\chi^{(\alpha,\beta)}}+N^{-1}|v|_{1,\chi^{(\alpha+1,\beta+1)}}\big). \tag{4.1}$$

Proof. By (2.13) and (2.25),

$$\begin{aligned}\|\mathcal{I}_{G,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)}}=\|\mathcal{I}_{G,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)},G,N}&=\sum_{j=0}^{N}v^2\big(\zeta^{(\alpha,\beta)}_{G,N,j}\big)\omega^{(\alpha,\beta)}_{G,N,j}\\&\leqslant cN^{-1}\sum_{j=0}^{N}v^2\big(\zeta^{(\alpha,\beta)}_{G,N,j}\big)\big(1-\zeta^{(\alpha,\beta)}_{G,N,j}\big)^{\alpha+1/2}\big(1+\zeta^{(\alpha,\beta)}_{G,N,j}\big)^{\beta+1/2}.\end{aligned}$$

Let $x=\cos\theta$ and $\widehat{v}(\theta)=v(\cos\theta)$. Then

$$\|\mathcal{I}_{G,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)}}\leqslant cN^{-1}\sum_{j=0}^{N}\widehat{v}^2\big(\theta^{(\alpha,\beta)}_{N,j}\big)\left(\sin\frac{\theta^{(\alpha,\beta)}_{N,j}}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta^{(\alpha,\beta)}_{N,j}}{2}\right)^{2\beta+1}.$$

According to theorem 8.9.1 of Szegö [38],

$$\theta^{(\alpha,\beta)}_{N,j}=\frac{1}{N+1}\big(j\pi+\mathcal{O}(1)\big),\quad 0\leqslant j\leqslant N, \tag{4.2}$$

where $\mathcal{O}(1)$ is bounded uniformly for all $0\leqslant j\leqslant N$. Now, let $a_0=\mathcal{O}(1)/(N+1)$ and $a_1=(N\pi+\mathcal{O}(1))/(N+1)$. Then $\theta^{(\alpha,\beta)}_{N,j}\in K_j\subset[a_0,a_1]$, K_j being of size $c/(N+1)$. Therefore

$$\|\mathcal{I}_{G,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)}}\leqslant cN^{-1}\sum_{j=0}^{N}\sup_{\theta\in K_j}\big|\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big|^2.$$

By using an inequality of space interpolation (see (13.7) of Bernardi and Maday [5]), we know that for any $f \in H^1(a,b)$,

$$\max_{a\leqslant x\leqslant b} \left|f(x)\right| \leqslant c\left(\frac{1}{\sqrt{b-a}}\|f\|_{L^2(a,b)} + \sqrt{b-a}\|\partial_x f\|_{L^2(a,b)}\right).$$

Hence

$$\begin{aligned}
\|\mathcal{I}_{G,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)}} &\leqslant c\sum_{j=0}^{N}\Big(\big\|\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big\|^2_{L^2(K_j)} \\
&\quad + N^{-2}\big\|\partial_\theta\big(\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big)\big\|^2_{L^2(K_j)}\Big) \\
&\leqslant c\Big(\big\|\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big\|^2_{L^2(0,\pi)} \\
&\quad + N^{-2}\big\|\partial_\theta\big(\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big)\big\|^2_{L^2(a_0,a_1)}\Big) \\
&\leqslant c\Big(\big\|\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big\|^2_{L^2(0,\pi)} \\
&\quad + N^{-2}\big\|\partial_\theta\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big\|^2_{L^2(0,\pi)} \\
&\quad + \sup_{a_0\leqslant\theta\leqslant a_1}\frac{1}{N^2\sin^2\theta}\cdot\big\|\widehat{v}(\theta)\big(\sin\tfrac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\tfrac{\theta}{2}\big)^{\beta+1/2}\big\|^2_{L^2(0,\pi)}\Big) \\
&\leqslant c\big(\|v\|^2_{\chi^{(\alpha,\beta)}} + N^{-2}\|\partial_x v\|^2_{\chi^{(\alpha+1,\beta+1)}}\big).
\end{aligned}$$

This yields (4.1). □

The above theorem leads to another stability result, which is very useful in numerical analysis of Jacobi pseudospectral method for nonlinear problems.

Theorem 4.2. Let $M, M' \in \mathbb{N}$. For any $\phi \in \mathcal{P}_M$ and $\psi \in \mathcal{P}_{M'}$,

$$\|\mathcal{I}_{G,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}} \leqslant c\left(1+\frac{M}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{4.3}$$

Moreover,

$$\left|(\phi,\psi)_{\chi^{(\alpha,\beta)},G,N}\right| \leqslant c\left(1+\frac{M}{N}\right)\left(1+\frac{M'}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}\|\psi\|_{\chi^{(\alpha,\beta)}}. \tag{4.4}$$

Proof. By (3.3) and (4.1),

$$\|\mathcal{I}_{G,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}} \leqslant c(\|\phi\|_{\chi^{(\alpha,\beta)}} + N^{-1}|\phi|_{1,\chi^{(\alpha+1,\beta+1)}}) \leqslant c\left(1+\frac{M}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}.$$

Next, by (2.25) and (4.3),

$$\begin{aligned}\left|(\phi,\psi)_{\chi^{(\alpha,\beta)},G,N}\right| &= \left|(\mathcal{I}_{G,N,\alpha,\beta}\phi,\mathcal{I}_{G,N,\alpha,\beta}\psi)_{\chi^{(\alpha,\beta)},G,N}\right| \\ &= \left|(\mathcal{I}_{G,N,\alpha,\beta}\phi,\mathcal{I}_{G,N,\alpha,\beta}\psi)_{\chi^{(\alpha,\beta)}}\right| \\ &\leqslant \|\mathcal{I}_{G,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}}\|\mathcal{I}_{G,N,\alpha,\beta}\psi\|_{\chi^{(\alpha,\beta)}} \\ &\leqslant c\left(1+\frac{M}{N}\right)\left(1+\frac{M'}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}\|\psi\|_{\chi^{(\alpha,\beta)}}. \qquad \square\end{aligned}$$

We now deal with Jacobi–Gauss interpolation approximation. Usually, we only consider the approximation in Sobolev spaces. But in many practical problems, it is more reasonable to deal with it in certain Hilbert spaces.

Theorem 4.3. If

$$\gamma \leqslant \alpha \leqslant \gamma+1, \qquad \delta \leqslant \beta \leqslant \delta+1, \tag{4.5}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\|\mathcal{I}_{G,N,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{2\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.6}$$

If, in addition, $\alpha=\gamma+1$ and $\beta=\delta+1$, then

$$\|\mathcal{I}_{G,N,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.7}$$

Proof. By virtue of (4.1) and lemma 3.10,

$$\begin{aligned}&\left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\gamma,\delta)}} \\ &\quad\leqslant c\left(\left\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\right\|_{\chi^{(\gamma,\delta)}} + N^{-1}\left|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\right|_{1,\chi^{(\gamma+1,\delta+1)}}\right) \\ &\quad\leqslant c\left(N^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*} + N^{-1}\left|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\right|_{1,\chi^{(\alpha,\beta)}}\right) \\ &\quad\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned}\|\mathcal{I}_{G,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} &\leqslant \left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\gamma,\delta)}} + \left\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\right\|_{\chi^{(\gamma,\delta)}} \\ &\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Furthermore, by (3.2), lemma 3.10 and (4.8),

$$\begin{aligned}&\left|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right|_{1,\chi^{(\alpha,\beta)}} \\ &\quad\leqslant cN^2\left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\alpha,\beta)}} \\ &\quad\leqslant cN^2\left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\gamma,\delta)}} \leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Therefore

$$
\begin{aligned}
|\mathcal{I}_{G,N,\gamma,\delta}v - v|_{1,\chi^{(\alpha,\beta)}} &\leqslant \left|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right|_{1,\chi^{(\alpha,\beta)}} + \left|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\right|_{1,\chi^{(\alpha,\beta)}} \\
&\leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}
$$

Then (4.6) comes from the previous statements and space interpolation.

We now prove (4.7). Thanks to $\alpha = \gamma + 1$ and $\beta = \delta + 1$, we have from (3.3), lemma 3.10 and (4.8) that

$$
\begin{aligned}
&\left|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant cN\left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\alpha-1,\beta-1)}} \\
&\quad \leqslant cN\left\|\mathcal{I}_{G,N,\gamma,\delta}v - P^1_{N,\alpha,\beta,\gamma,\delta}v\right\|_{\chi^{(\gamma,\delta)}} \leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \qquad (4.9)
\end{aligned}
$$

Therefore, (4.7) follows from the same argument as in the previous paragraph. □

Remark 4.1. For any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$
\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{2\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \qquad (4.10)
$$

We now give another main result on the Jacobi–Gauss interpolation approximation, in which we use different interpolation points from those in the last theorem.

Theorem 4.4. If (3.9) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $r \geqslant 1$,

$$
\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \qquad (4.11)
$$

Proof. (3.8) with (3.9) implies that

$$
\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}}.
$$

So it suffices to estimate $|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}}$. In fact, we have from (3.2), (3.22) and (4.10) that

$$
\begin{aligned}
&|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant \left|P^1_{N,\alpha,\beta}v - v\right|_{1,\chi^{(\alpha,\beta)}} + \left|\mathcal{I}_{G,N,\alpha,\beta}v - P^1_{N,\alpha,\beta}v\right|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant \left\|P^1_{N,\alpha,\beta}v - v\right\|_{1,\chi^{(\alpha,\beta)}} + cN^2\left\|\mathcal{I}_{G,N,\alpha,\beta}v - P^1_{N,\alpha,\beta}v\right\|_{\chi^{(\alpha,\beta)}} \\
&\quad \leqslant c\left(N^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*} + N^2\left(\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} + \left\|P^1_{N,\alpha,\beta}v - v\right\|_{\chi^{(\alpha,\beta)}}\right)\right) \\
&\quad \leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}
$$

This leads to (4.11). □

We next deal with Jacobi–Gauss–Radau interpolation approximation. As usual, the first result is related to the stability of the interpolation.

Theorem 4.5. For any $v \in {}_0H^1_{\chi^{(\alpha+1,\beta+1)}}(\Lambda) \cap L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\|\mathcal{I}_{R,N,\alpha,\beta} v\|_{\chi^{(\alpha,\beta)}} \leqslant c\big(\|v\|_{\chi^{(\alpha,\beta)}} + N^{-1}|v|_{1,\chi^{(\alpha+1,\beta+1)}}\big). \tag{4.12}$$

Proof. By (2.18) and (2.25),

$$\begin{aligned}\|\mathcal{I}_{R,N,\alpha,\beta} v\|^2_{\chi^{(\alpha,\beta)}} &= \|\mathcal{I}_{R,N,\alpha,\beta} v\|^2_{\chi^{(\alpha,\beta)},R,N} = \sum_{j=0}^{N} v^2\big(\zeta^{(\alpha,\beta)}_{G,N,j}\big)\omega^{(\alpha,\beta)}_{G,N,j} \\ &\leqslant cN^{-1}\sum_{j=0}^{N-1} v^2\big(\zeta^{(\alpha,\beta)}_{G,N,j}\big)\big(1-\zeta^{(\alpha,\beta)}_{G,N,j}\big)^{\alpha+1/2}\big(1+\zeta^{(\alpha,\beta)}_{G,N,j}\big)^{\beta+1/2}.\end{aligned}$$

Then (4.12) follows from (2.16), (4.2) and the same strategies as in the proof of theorem 4.1. □

The following result also plays an important role in numerical analysis of Jacobi pseudospectral method for nonlinear problems.

Theorem 4.6. Let $M, M' \in \mathbb{N}$. For any $\phi \in {}_0\mathcal{P}_M$ and $\psi \in {}_0\mathcal{P}_{M'}$,

$$\|\mathcal{I}_{R,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}} \leqslant c\left(1+\frac{M}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{4.13}$$

Moreover,

$$\big|(\phi,\psi)_{\chi^{(\alpha,\beta)},R,N}\big| \leqslant c\left(1+\frac{M}{N}\right)\left(1+\frac{M'}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}\|\psi\|_{\chi^{(\alpha,\beta)}}. \tag{4.14}$$

Proof. By (3.3) and (4.12),

$$\|\mathcal{I}_{R,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}} \leqslant c\big(\|\phi\|_{\chi^{(\alpha,\beta)}} + N^{-1}|\phi|_{1,\chi^{(\alpha+1,\beta+1)}}\big) \leqslant c\left(1+\frac{M}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}.$$

Next, by (2.25) and the above fact,

$$\begin{aligned}\big|(\phi,\psi)_{\chi^{(\alpha,\beta)},R,N}\big| &= \big|(\mathcal{I}_{R,N,\alpha,\beta}\phi, \mathcal{I}_{R,N,\alpha,\beta}\psi)_{\chi^{(\alpha,\beta)},R,N}\big| \\ &= \big|(\mathcal{I}_{R,N,\alpha,\beta}\phi, \mathcal{I}_{R,N,\alpha,\beta}\psi)_{\chi^{(\alpha,\beta)}}\big| \\ &\leqslant \|\mathcal{I}_{R,N,\alpha,\beta}\phi\|_{\chi^{(\alpha,\beta)}}\|\mathcal{I}_{R,N,\alpha,\beta}\psi\|_{\chi^{(\alpha,\beta)}} \\ &\leqslant c\left(1+\frac{M}{N}\right)\left(1+\frac{M'}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}\|\psi\|_{\chi^{(\alpha,\beta)}}\end{aligned}$$

which completes the proof. □

We now estimate the difference between v and $\mathcal{I}_{R,N,\alpha,\beta}v$ in the weighted Sobolev space $H^{\mu}_{\chi^{(\alpha,\beta)}}(\Lambda),\ 0 \leqslant \mu \leqslant 1$.

Theorem 4.7. If (3.30) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{2\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.15}$$

Proof. The space interpolation allows us to only consider the cases $\mu = 0, 1$. Let $\widehat{P}^1_{N,\alpha,\beta}$ be the same as in lemma 3.13. Since $\mathcal{I}_{R,N,\alpha,\beta}(\widehat{P}^1_{N,\alpha,\beta}v)$ coincides with $\widehat{P}^1_{N,\alpha,\beta}v$, we have that

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \leqslant \big\|\widehat{P}^1_{N,\alpha,\beta}v - v\big\|_{\chi^{(\alpha,\beta)}} + \big\|\mathcal{I}_{R,N,\alpha,\beta}(v - \widehat{P}^1_{N,\alpha,\beta}v)\big\|_{\chi^{(\alpha,\beta)}}. \tag{4.16}$$

Furthermore, by (3.2),

$$\begin{aligned}
&|\mathcal{I}_{R,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant \big\|\widehat{P}^1_{N,\alpha,\beta}v - v\big\|_{1,\chi^{(\alpha,\beta)}} + \big|\mathcal{I}_{R,N,\alpha,\beta}(v - \widehat{P}^1_{N,\alpha,\beta}v)\big|_{1,\chi^{(\alpha,\beta)}} \\
&\quad \leqslant c\Big(\big\|\widehat{P}^1_{N,\alpha,\beta}v - v\big\|_{1,\chi^{(\alpha,\beta)}} + N^2\big\|\mathcal{I}_{R,N,\alpha,\beta}(v - \widehat{P}^1_{N,\alpha,\beta}v)\big\|_{\chi^{(\alpha,\beta)}}\Big).
\end{aligned} \tag{4.17}$$

We can use (3.32) to estimate the first term at the right side of (4.17). Thus it suffices to estimate the second one. By (3.32) and (4.12),

$$\begin{aligned}
\big\|\mathcal{I}_{R,N,\alpha,\beta}\big(v - \widehat{P}^1_{N,\alpha,\beta}v\big)\big\|_{\chi^{(\alpha,\beta)}} &\leqslant c\Big(\big\|\widehat{P}^1_{N,\alpha,\beta}v - v\big\|_{\chi^{(\alpha,\beta)}} + N^{-1}\big|\widehat{P}^1_{N,\alpha,\beta}v - v\big|_{1,\chi^{(\alpha+1,\beta+1)}}\Big) \\
&\leqslant c\Big(\big\|\widehat{P}^1_{N,\alpha,\beta}v - v\big\|_{\chi^{(\alpha,\beta)}} + N^{-1}\big|\widehat{P}^1_{N,\alpha,\beta}v - v\big|_{1,\chi^{(\alpha,\beta)}}\Big) \\
&\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned} \tag{4.18}$$

The combination of (4.16)–(4.18) leads to (4.15). □

We can also deal with the Jacobi–Gauss–Radau interpolation approximation in the Hilbert space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \leqslant \mu \leqslant 1$, as in the following theorem.

Theorem 4.8. If one of the following conditions holds:

$$\text{(i)} \quad \gamma \leqslant \alpha \leqslant \gamma + 1, \quad \delta \leqslant \beta \leqslant \delta + 1, \quad 0 < \alpha < 1, \quad \beta < 1, \tag{4.19}$$

$$\text{(ii)} \quad \gamma \leqslant \alpha \leqslant \gamma + 1, \quad \beta = \delta = 0, \tag{4.20}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\|\mathcal{I}_{R,N,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{2\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.21}$$

If, in addition, $\alpha = \gamma + 1$ and $\beta = \delta + 1$, then

$$\|\mathcal{I}_{R,N,\gamma,\delta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.22}$$

Proof. We first prove (4.21). By (4.12), (3.27), (4.19) and (4.20), we find that

$$\begin{aligned}
&\big\|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big\|_{\chi^{(\gamma,\delta)}} \\
&\quad \leqslant c\Big(\big\|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big\|_{\chi^{(\gamma,\delta)}} + N^{-1}\big|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big|_{1,\chi^{(\gamma+1,\delta+1)}}\Big)
\end{aligned}$$

$$\leqslant c\big(\big\|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big\|_{\chi^{(\gamma,\delta)}} + N^{-1}\big|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta} - v\big|_{1,\chi^{(\alpha,\beta)}}\big)$$
$$\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.23}$$

Using (3.27) again yields that

$$\|\mathcal{I}_{R,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant \big\|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big\|_{\chi^{(\gamma,\delta)}} + \big\|\mathcal{I}_{R,N,\gamma,\delta}(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v)\big\|_{\chi^{(\gamma,\delta)}}$$
$$\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.24}$$

Furthermore, the previous statements and (3.2) imply that for $\alpha \geqslant \gamma$ and $\beta \geqslant \delta$,

$$\big|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big|_{1,\chi^{(\alpha,\beta)}}$$
$$\leqslant cN^2\big\|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big\|_{\chi^{(\alpha,\beta)}}$$
$$\leqslant cN^2\big\|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big\|_{\chi^{(\gamma,\delta)}} \leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

This fact and (3.27) lead to

$$|\mathcal{I}_{R,N,\gamma,\delta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant \big|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big|_{1,\chi^{(\alpha,\beta)}} + \big|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big|_{1,\chi^{(\alpha,\beta)}}$$
$$\leqslant cN^{2-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.25}$$

Then, (4.21) follows from (4.24), (4.25) and space interpolation.

If, in addition, $\alpha = \gamma + 1$ and $\beta = \delta + 1$, then we get from (3.3) and (4.23) that

$$\big|\mathcal{I}_{R,N,\gamma,\delta}(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v)\big|_{1,\chi^{(\alpha,\beta)}} \leqslant cN\big\|\mathcal{I}_{R,N,\gamma,\delta}(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v)\big\|_{\chi^{(\gamma,\delta)}}$$
$$\leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Thus by (3.27),

$$|\mathcal{I}_{R,N,\gamma,\delta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant \big|\widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v - v\big|_{1,\chi^{(\alpha,\beta)}} + \big|\mathcal{I}_{R,N,\gamma,\delta}\big(v - \widehat{P}^1_{N,\alpha,\beta,\gamma,\delta}v\big)\big|_{1,\chi^{(\alpha,\beta)}}$$
$$\leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.$$

Finally, we reach (4.22) by the above facts and space interpolation. □

In the end of this section, we consider the Jacobi–Gauss–Lobatto interpolation approximation. The first result is also related to the stability of interpolation.

Theorem 4.9. For any $v \in H^1_{0,\chi^{(\alpha+1,\beta+1)}}(\Lambda) \cap L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}v\|_{\chi^{(\alpha,\beta)}} \leqslant c\big(\|v\|_{\chi^{(\alpha,\beta)}} + N^{-1}|v|_{1,\chi^{(\alpha+1,\beta+1)}}\big). \tag{4.26}$$

Moreover, for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap L^2_{\chi^{(\alpha-1,\beta-1)}}(\Lambda)$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}v\|_{\chi^{(\alpha-1,\beta-1)}} \leqslant c\big(\|v\|_{\chi^{(\alpha-1,\beta-1)}} + N^{-1}|v|_{1,\chi^{(\alpha,\beta)}}\big). \tag{4.27}$$

Proof. By (2.23) and (2.26),

$$\|\mathcal{I}_{L,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)}} \leqslant c\|\mathcal{I}_{L,N,\alpha,\beta}v\|^2_{\chi^{(\alpha,\beta)},L,N} = c\sum_{j=0}^{N} v^2\big(\zeta^{(\alpha,\beta)}_{L,N,j}\big)\omega^{(\alpha,\beta)}_{L,N,j}$$
$$\leqslant cN^{-1}\sum_{j=1}^{N-1} v^2\big(\zeta^{(\alpha,\beta)}_{L,N,j}\big)\big(1-\zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\alpha+1/2}\big(1+\zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\beta+1/2}.$$

Thus we reach (4.26) by (2.21), (4.2) and the same strategies as in the proof of theorem 4.1.

We now prove (4.27). Clearly, $(\mathcal{I}_{L,N,\alpha,\beta}v(x))^2(1-x^2)^{-1} \in \mathcal{P}_{2N-2}$. So by (2.23) and (2.25),

$$\|\mathcal{I}_{L,N,\alpha,\beta}v\|^2_{\chi^{(\alpha-1,\beta-1)}} = \big\|\mathcal{I}_{L,N,\alpha,\beta}v\chi^{(-1/2,-1/2)}\big\|^2_{\chi^{(\alpha,\beta)},L,N}$$
$$\leqslant cN^{-1}\sum_{j=1}^{N-1} v^2\big(\zeta^{(\alpha,\beta)}_{L,N,j}\big)\big(1-\zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\alpha-1/2}\big(1+\zeta^{(\alpha,\beta)}_{L,N,j}\big)^{\beta-1/2}.$$

The rest part of the proof is similar to the proof of theorem 4.1. □

Theorem 4.10. Let $M, M' \in \mathbb{N}$. For any $\phi \in \mathcal{P}^0_M$, $\psi \in \mathcal{P}^0_{M'}$ and $\mu = 0, 1$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}\phi\|_{\chi^{(\alpha-\mu,\beta-\mu)}} \leqslant c\left(1+\frac{M}{N}\right)\|\phi\|_{\chi^{(\alpha-\mu,\beta-\mu)}}. \tag{4.28}$$

Moreover,

$$\big|(\phi,\psi)_{\chi^{(\alpha,\beta)},L,N}\big| \leqslant c\left(1+\frac{M}{N}\right)\left(1+\frac{M'}{N}\right)\|\phi\|_{\chi^{(\alpha,\beta)}}\|\psi\|_{\chi^{(\alpha,\beta)}}. \tag{4.29}$$

Proof. The result (4.28) with $\mu = 0$ follows from (3.4) and (4.26). While (4.28) with $\mu = 1$ comes from (3.4) and (4.27). We can prove (4.29) by a similar argument as in the derivation of (4.14). □

We now turn to the main result on the Jacobi–Gauss–Lobatto interpolation approximation.

Theorem 4.11. If $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{2\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.30}$$

In particular, if $0 < \alpha, \beta < 1$, then

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)}} \leqslant cN^{\mu-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \tag{4.31}$$

Proof. Let $\widetilde{P}^1_{N,\alpha,\beta}$ be the same as in (3.61). Since $\widetilde{P}^1_{N,\alpha,\beta}v(x) = \mathcal{I}_{L,N,\alpha,\beta}v(x) = v(x)$ for $|x| = 1$, we can apply (4.26) to $\mathcal{I}_{L,N,\alpha,\beta}v - \widetilde{P}^1_{N,\alpha,\beta}v$. So by (3.60),

$$\begin{aligned}\left\|\mathcal{I}_{L,N,\alpha,\beta}v - \widetilde{P}^1_{N,\alpha,\beta}v\right\|_{\chi^{(\alpha,\beta)}} &\leqslant c\left(\left\|\widetilde{P}^1_{N,\alpha,\beta}v - v\right\|_{\chi^{(\alpha,\beta)}} + N^{-1}\left|\widetilde{P}^1_{N,\alpha,\beta} - v\right|_{1,\chi^{(\alpha+1,\beta+1)}}\right)\\ &\leqslant cN^{-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Replacing $\mathcal{I}_{R,N,\alpha,\beta}$ and $\widehat{P}^1_{N,\alpha,\beta}$ in (4.16)–(4.18) by $\mathcal{I}_{L,N,\alpha,\beta}$ and $\widetilde{P}^1_{N,\alpha,\beta}$, respectively, we reach (4.30) by using (3.3), (3.60) and a similar argument as in the proof of theorem 4.7.

We now prove (4.31). Since $0 < \alpha, \beta < 1$, we deduce from (3.4), (4.27) and (3.60) that

$$\begin{aligned}\left|\mathcal{I}_{L,N,\alpha,\beta}v - \widetilde{P}^1_{N,\alpha,\beta}v\right|_{1,\chi^{(\alpha,\beta)}} &\leqslant cN\left\|\mathcal{I}_{L,N,\alpha,\beta}\big(v - \widetilde{P}^1_{N,\alpha,\beta}v\big)\right\|_{\chi^{(\alpha-1,\beta-1)}}\\ &\leqslant c\left(N\left\|\widetilde{P}^1_{N,\alpha,\beta}v - v\right\|_{\chi^{(\alpha-1,\beta-1)}} + \left|\widetilde{P}^1_{N,\alpha,\beta}v - v\right|_{1,\chi^{(\alpha,\beta)}}\right)\\ &\leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Thus, by remark 3.2 and (3.60),

$$\begin{aligned}\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{1,\chi^{(\alpha,\beta)}} &\leqslant c\left(\left|\mathcal{I}_{L,N,\alpha,\beta}v - \widetilde{P}^1_{N,\alpha,\beta}v\right|_{1,\chi^{(\alpha,\beta)}} + \left|\widetilde{P}^1_{N,\alpha,\beta}v - v\right|_{1,\chi^{(\alpha,\beta)}}\right)\\ &\leqslant cN^{1-r}\|v\|_{r,\chi^{(\alpha,\beta)},*}. \qquad (4.32)\end{aligned}$$

Eventually, (4.31) follows from (4.30) with $\mu = 0$, (4.32) and space interpolation. □

5. Applications

This section is for Jacobi pseudospectral method for singular differential equations. We first consider the following equation:

$$\begin{cases}-\partial_x(a(x)\partial_x U(x)) + c(x)U(x) = f(x), & x \in \Lambda,\\ a(x)\partial_x U(x) \to 0, & \text{as } |x| \to 1,\end{cases} \qquad (5.1)$$

where $a(x) \geqslant 0$, $c(x) \geqslant 0$ and $f(x)$ are given functions, and $a(x)$ and $c(x)$ degenerate as $|x| \to 1$. Such problems appear in many fields, see Ames [2], Chamber [9], and Keller [31]. Also there are many literatures concerning their numerical simulations, see, e.g., Jesperson [27], Eriksson and Thomee [15], Elgebeily and Abuzaid [13], and Schreiber [35]. Here, we use Jacobi pseudospectral method to resolve (5.1) numerically.

For simplicity, assume that

$$a(x) = a_1(x)\chi^{(\alpha,\beta)}(x), \qquad c(x) = c_1(x)\chi^{(\gamma,\delta)}(x), \qquad (5.2)$$

$$a_1(x) \in H^s(\Lambda), \qquad c_1(x) \in H^{s'}(\Lambda), \qquad s, s' > \frac{3}{4}, \qquad (5.3)$$

and for $x \in \overline{\Lambda}$,

$$a_1(x) \geqslant a_{\min} > 0, \qquad c_1(x) \geqslant c_{\min} > 0. \qquad (5.4)$$

Let

$$b_{\alpha,\beta,\gamma,\delta}(u,v) = (a_1\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (c_1 u, v)_{\chi^{(\gamma,\delta)}}.$$

A weak formulation of (5.1) is to find $U \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ such that

$$b_{\alpha,\beta,\gamma,\delta}(U,v) = (f,v), \quad \forall v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{5.5}$$

Now, let N be any positive even number, $\widetilde{a}_1 = P_{N/2,0,0,0,0}a_1$ and $\widetilde{c}_1 = P_{N/2,0,0,0,0}c_1$. Further, let $\widetilde{f}(x) = \chi^{(-\gamma,-\delta)}(x)f(x)$. The numerical scheme for (5.5) is to find $u_N \in \mathcal{P}_N$ such that

$$\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(u_N,\phi) = \big(\widetilde{f},\phi\big)_{\chi^{(\gamma,\delta)},G,N}, \quad \forall \phi \in \mathcal{P}_N, \tag{5.6}$$

where

$$\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(u,v) = (\widetilde{a}_1\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)},G,N} + (\widetilde{c}_1 u, v)_{\chi^{(\gamma,\delta)},G,N}.$$

We know from (2.25) that for any $\phi, \psi \in \mathcal{P}_N$,

$$\begin{aligned}\big|\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)\big| &\leqslant \|\widetilde{a}_1\|_\infty \|\partial_x\phi\|_{\chi^{(\alpha,\beta)},G,N}\|\partial_x\psi\|_{\chi^{(\alpha,\beta)},G,N} \\ &\quad + \|\widetilde{c}_1\|_\infty\|\phi\|_{\chi^{(\gamma,\delta)},G,N}\|\psi\|_{\chi^{(\gamma,\delta)},G,N} \\ &\leqslant \big(\|\widetilde{a}_1\|_\infty + \|\widetilde{c}_1\|_\infty\big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}$$

Moreover, we have from the property of the Legendre approximation (see Canuto, Hussaini, Quarteroni and Zang [8]), that for any $v \in H^r(\Lambda)$ with $r \geqslant 3/4$,

$$\|P_{N,0,0,0,0}v - v\|_\infty \leqslant cN^{3/4-r}\|v\|_r. \tag{5.7}$$

The above inequality with the imbedding theorem implies that

$$\|\widetilde{a}_1\|_\infty \leqslant c\|a_1\|_{3/4}, \qquad \|\widetilde{c}_1\|_\infty \leqslant c\|c_1\|_{3/4}.$$

Therefore

$$\big|\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)\big| \leqslant c\big(\|a_1\|_{3/4} + \|c_1\|_{3/4}\big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}. \tag{5.8}$$

On the other hand, we have from (5.3), (5.4), (5.7) and (2.25) that for $s, s' > 3/4$, large N and any $\phi \in \mathcal{P}_N$,

$$\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(\phi,\phi) \geqslant c\|\phi\|^2_{1,\alpha,\beta,\gamma,\delta}. \tag{5.9}$$

Hence, by the Lax–Milgram lemma, (5.6) has a unique solution such that

$$\|u_N\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c\big\|\mathcal{I}_{G,N,\gamma,\delta}\widetilde{f}\big\|_{\chi^{(\gamma,\delta)}}.$$

Now, we compare the numerical solution u_N with the exact solution U.

Theorem 5.1. Assume that

(i) $\alpha \leqslant \gamma + 2,\ \beta \leqslant \delta + 2$,

(ii) $a_1 \in H^s(\Lambda),\ c_1 \in H^{s'}(\Lambda)$ with $s, s' > 3/4$,

(iii) $U \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ with $r \geqslant 1$, and $\widetilde{f} \in H^\sigma_{\chi^{(\gamma,\delta)},*}(\Lambda)$ with $\sigma \geqslant 1$.

Then

$$\|u_N - U\|_{1,\alpha,\beta,\gamma,\delta} \leqslant d\left(\frac{N}{2}\right)^{1-r} \|U\|_{r,\chi^{(\alpha,\beta)},*} + c\left(\left(\frac{N}{2}\right)^{3/4-s} \|a_1\|_s |U|_{1,\chi^{(\alpha,\beta)}} + \left(\frac{N}{2}\right)^{3/4-s'} \|c_1\|_{s'} \|U\|_{\chi^{(\gamma,\delta)}} + N^{-\sigma} \|\widetilde{f}\|_{\sigma,\chi^{(\gamma,\delta)},*}\right), \quad (5.10)$$

where d is a positive constant depending only on the norms $\|a_1\|_{3/4}$ and $\|c_1\|_{3/4}$.

Proof. To obtain better error estimations, let $U_N = P^1_{N,\alpha,\beta,\gamma,\delta}U$. By (5.5), (5.6) and the ellipticity (5.9),

$$\begin{aligned} c\|u_N - U_N\|^2_{1,\alpha,\beta,\gamma,\delta} &\leqslant \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(u_N - U_N, u_N - U_N) \\ &= \big(\widetilde{f}, u_N - U_N\big)_{\chi^{(\gamma,\delta)},G,N} - \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N, u_N - U_N) \\ &= b_{\alpha,\beta,\gamma,\delta}(U, u_N - U_N) - \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N, u_N - U_N) \\ &\quad + \big(\widetilde{f}, u_N - U_N\big)_{\chi^{(\gamma,\delta)},G,N} - \big(\widetilde{f}, u_N - U_N\big)_{\chi^{(\gamma,\delta)}}. \end{aligned} \quad (5.11)$$

Thus

$$\|u_N - U_N\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c\left(\sup_{\phi\in\mathcal{P}_N,\phi\neq 0} \frac{|b_{\alpha,\beta,\gamma,\delta}(U,\phi) - \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)|}{\|\phi\|_{1,\alpha,\beta,\gamma,\delta}} + \sup_{\phi\in\mathcal{P}_N,\phi\neq 0} \frac{|(\widetilde{f},\phi)_{\chi^{(\gamma,\delta)}} - (\widetilde{f},\phi)_{\chi^{(\gamma,\delta)},G,N}|}{\|\phi\|_{1,\alpha,\beta,\gamma,\delta}}\right). \quad (5.12)$$

We now estimate the first term at the right side of (5.12). For simplicity, let

$$\widetilde{b}_{\alpha,\beta,\gamma,\delta}(u, v) = (\widetilde{a}_1 \partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (\widetilde{c}_1 u, v)_{\chi^{(\gamma,\delta)}}$$

and

$$b_{\alpha,\beta,\gamma,\delta}(U,\phi) - \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi) = G_1 + G_2,$$

where

$$\begin{aligned} G_1 &= b_{\alpha,\beta,\gamma,\delta}(U,\phi) - \widetilde{b}_{\alpha,\beta,\gamma,\delta}(U,\phi), \\ G_2 &= \widetilde{b}_{\alpha,\beta,\gamma,\delta}(U,\phi) - \widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi). \end{aligned}$$

By (5.7),

$$\begin{aligned} |G_1| &\leqslant \big(\|\widetilde{a}_1 - a_1\|_\infty |U|_{1,\chi^{(\alpha,\beta)}} + \|\widetilde{c}_1 - c_1\|_\infty \|U\|_{\chi^{(\gamma,\delta)}}\big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta} \\ &\leqslant c\left(\left(\frac{N}{2}\right)^{3/4-s} \|a_1\|_s |U|_{1,\chi^{(\alpha,\beta)}} + \left(\frac{N}{2}\right)^{3/4-s'} \|c_1\|_{s'} \|U\|_{\chi^{(\gamma,\delta)}}\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}. \end{aligned} \quad (5.13)$$

On the other hand, by (2.25),

$$\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}\big(P^1_{N/2,\alpha,\beta,\gamma,\delta}U, \phi\big) = \widetilde{b}_{\alpha,\beta,\gamma,\delta}\big(P^1_{N/2,\alpha,\beta,\gamma,\delta}U, \phi\big), \quad \forall \phi \in \mathcal{P}_N.$$

So

$$|G_2| \leqslant \left|\widetilde{b}_{\alpha,\beta,\gamma,\delta}\left(P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U, \phi\right)\right| + \left|\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}\left(P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U_N, \phi\right)\right|.$$

According to (5.8),

$$\begin{aligned}
&\left|\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}\left(P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U_N, \phi\right)\right| \\
&\quad\leqslant d\left\| P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U_N\right\|_{1,\alpha,\beta,\gamma,\delta}\|\phi\|_{1,\alpha,\beta,\gamma,\delta} \\
&\quad\leqslant d\left(\left\| P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U\right\|_{1,\alpha,\beta,\gamma,\delta} + \|U_N - U\|_{1,\alpha,\beta,\gamma,\delta}\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.
\end{aligned}$$

Using lemma 3.10, we get that

$$\begin{aligned}
|G_2| &\leqslant d\left(\left\| P^1_{N/2,\alpha,\beta,\gamma,\delta}U - U\right\|_{1,\alpha,\beta,\gamma,\delta} + \|U_N - U\|_{1,\alpha,\beta,\gamma,\delta}\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta} \\
&\leqslant d\left(\frac{N}{2}\right)^{1-r}\|U\|_{r,\chi^{(\alpha,\beta)},*}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.
\end{aligned} \tag{5.14}$$

Next, we use (4.10) to obtain that

$$\begin{aligned}
\left|(f,\phi) - \left(\widetilde{f},\phi\right)_{\chi^{(\gamma,\delta)},G,N}\right| &\leqslant \left\|\mathcal{I}_{G,N,\gamma,\delta}\widetilde{f} - \widetilde{f}\right\|_{\chi^{(\gamma,\delta)}}\|\phi\|_{\chi^{(\gamma,\delta)}} \\
&\leqslant cN^{-\sigma}\left\|\widetilde{f}\right\|_{\sigma,\chi^{(\gamma,\delta)},*}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.
\end{aligned} \tag{5.15}$$

A combination of (5.12)–(5.15) leads to (5.10). □

Remark 5.1. If $a_1 \in \mathcal{P}_{k+2}$, $c_1 \in \mathcal{P}_k$, $k \geqslant 0$, and (5.4) holds, then we can approximate (5.5) in another way. In this case, let

$$b_{\alpha,\beta,\gamma,\delta,N}(u,v) = (a_1\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)},G,N} + (c_1 u, v)_{\chi^{(\gamma,\delta)},G,N}, \quad \forall u,v \in C^1(\Lambda).$$

The corresponding numerical scheme is to find $u_N \in \mathcal{P}_N$ such that

$$b_{\alpha,\beta,\gamma,\delta,N}(u_N,\phi) = \left(\widetilde{f},\phi\right)_{\chi^{(\gamma,\delta)},G,N}, \quad \forall\phi \in \mathcal{P}_N.$$

It can be verified that for $\phi, \psi \in \mathcal{P}_N$,

$$\left|b_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)\right| \leqslant c\left(\|a_1\|_\infty + \|c_1\|_\infty\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}.$$

Also by (5.4), for any $\phi \in \mathcal{P}_N$,

$$b_{\alpha,\beta,\gamma,\delta,N}(\phi,\phi) \geqslant c\|\phi\|^2_{1,\alpha,\beta,\gamma,\delta}.$$

Therefore, we can derive estimates like (5.11) and (5.12). But $\widetilde{b}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)$ is now replaced by $b_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)$. Thanks to (2.25),

$$b_{\alpha,\beta,\gamma,\delta}\left(P^1_{N-k,\alpha,\beta,\gamma,\delta}U,\phi\right) = b_{\alpha,\beta,\gamma,\delta,N}\left(P^1_{N-k,\alpha,\beta,\gamma,\delta}U,\phi\right), \quad \forall\phi \in \mathcal{P}_N. \tag{5.16}$$

If conditions (i) and (iii) of theorem 5.1 are fulfilled, then by (5.16) and an argument similar to the derivation of (5.14), we assert that

$$
\begin{aligned}
&\big|b_{\alpha,\beta,\gamma,\delta}(U,\phi)-b_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)\big|\\
&\quad\leqslant \big|b_{\alpha,\beta,\gamma,\delta}\big(P^1_{N-k,\alpha,\beta,\gamma,\delta}U-U,\phi\big)\big|+\big|b_{\alpha,\beta,\gamma,\delta,N}\big(P^1_{N-k,\alpha,\beta,\gamma,\delta}U-U_N,\phi\big)\big|\\
&\quad\leqslant cN^{1-r}\|U\|_{r,\chi^{(\alpha,\beta)},*}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.
\end{aligned}
$$

In addition, (5.15) holds too. Finally, we conclude that

$$
\|u_N-U\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\big(N^{1-r}\|U\|_{r,\chi^{(\alpha,\beta)},*}+N^{-\sigma}\|\widetilde{f}\|_{\sigma,\chi^{(\gamma,\delta)},*}\big). \tag{5.17}
$$

We next take the Fisher-like equation as an example to show how to deal with nonlinear problems. We consider the problem

$$
\begin{cases}
\partial_t U(x,t)-\partial_x\big(w(x)\partial_x U(x,t)\big)=U(x,t)\big(1-U(x,t)\big)+f(x,t), & x\in\Lambda,\ t\in(0,T],\\
U(-1,t)=\lim\limits_{x\to 1}w(x)\partial_x U(x,t)=0, & t\in[0,T],\\
U(x,0)=U_0(x), & x\in\overline{\Lambda},
\end{cases}
\tag{5.18}
$$

where $w(x)$, $U_0(x)$ and $f(x,t)$ are given functions. The coefficient $w(x)$ degenerates as $x\to 1$. For simplicity, suppose that $w(x)=(1-x)^{\alpha}$. Let

$$
a_\alpha\big(U(t),v\big)=\big(\partial_x U(t),\partial_x v\big)_{\chi^{(\alpha,0)}}+\big(U(t),v\big).
$$

A weak formulation of (5.18) is to find $u_N\in L^2(0,T;{}_0H^1_{\alpha,0,0,0}(\Lambda))$ such that

$$
\begin{cases}
\big(\partial_t U(t),v\big)+a_\alpha\big(U(t),v\big)=\big(2U(t)-U^2(t),v\big)+\big(f(t),v\big),\\
\forall v\in {}_0H^1_{\alpha,0,0,0}(\Lambda),\quad t\in(0,T],\\
U(0)=U_0.
\end{cases}
\tag{5.19}
$$

If $f\in L^2(0,T;({}_0H^1_{\alpha,0,0,0}(\Lambda))')$ and $U_0\in L^2(\Lambda)$ then (5.19) has a unique solution.

For simplicity, let $\mathcal{R}_N=\mathcal{I}_{R,N,0,0}$, $(\cdot,\cdot)_N=(\cdot,\cdot)_{\chi^{(0,0)},R,N}$ and $\|\cdot\|_N=\|\cdot\|_{\chi^{(0,0)},R,N}$. Let u_N be the approximation of U. The numerical scheme for (5.19) is to find $u_N(t)\in {}_0\mathcal{P}_N$ such that for all $0\leqslant t\leqslant T$,

$$
\begin{cases}
\big(\partial_t u_N(t),\phi\big)_N+a_{\alpha,N}\big(u_N(t),\phi\big)=\big(2u_N(t)-u_N^2(t),\phi\big)_N+\big(f(t),\phi\big)_N,\\
\forall\phi\in {}_0\mathcal{P}_N,\ t\in(0,T],\\
u_N(0)=u_{N,0}=\mathcal{R}_N U_0,
\end{cases}
\tag{5.20}
$$

where

$$
a_{\alpha,N}\big(u_N(t),v\big)=\big(\partial_x u_N(t),\partial_x v\big)_{\chi^{(\alpha,0)},R,N}+\big(u_N(t),v\big)_N.
$$

We now analyze the stability of scheme (5.20). Since it is a nonlinear problem, it is not possible to possess the stability in the sense of Courant et al. [11], also see Richtmeyer and Morton [34]. But it might be stable in the sense of Guo [17]. Suppose that $u_{N,0}$ and f have the errors $\widetilde{u}_{N,0}$ and $\widetilde{f}$, respectively, which induce the error of u_N, denoted by $\widetilde{u}_N$. By (5.20), we get that for any $\phi \in {}_0\mathcal{P}_N$ and $t \in (0, T]$,

$$\begin{cases} (\partial_t \widetilde{u}_N(t), \phi)_N + a_{\alpha,N}\big(\widetilde{u}_N(t), \phi\big) \\ \quad = \big(2\widetilde{u}_N(t) - 2u_N(t)\widetilde{u}_N(t) - \widetilde{u}_N^2(t), \phi\big)_N + \big(\widetilde{f}(t), \phi\big)_N, \\ \widetilde{u}_N(0) = \widetilde{u}_{N,0}. \end{cases} \tag{5.21}$$

Taking $\phi = \widetilde{u}_N(t)$ in (5.21), we get from (2.25) that

$$\frac{1}{2}\partial_t \big\|\widetilde{u}_N(t)\big\|^2 + \big\|\widetilde{u}_N(t)\big\|^2_{1,\alpha,0,0,0} \leqslant \big|F(t, \widetilde{u}_N)\big|, \tag{5.22}$$

where

$$F(t, \widetilde{u}_N) = 2\big\|\widetilde{u}_N(t)\big\|^2 - \big(2u_N(t)\widetilde{u}_N(t) + \widetilde{u}_N^2(t), \widetilde{u}_N(t)\big)_N + \big(\widetilde{f}(t), \widetilde{u}_N(t)\big)_N.$$

We now estimate $|F(t, \widetilde{u}_N)|$. By (2.25), we have that

$$\big|\big(u_N(t)\widetilde{u}_N(t), \widetilde{u}_N(t)\big)_N\big| \leqslant |||u_N|||_\infty \|\widetilde{u}_N(t)\|^2, \tag{5.23}$$

where $|||u_N|||_\infty = \max_{0\leqslant t\leqslant T} \|u_N(t)\|_\infty$. Similarly,

$$\big|\big(\widetilde{f}(t), \widetilde{u}_N(t)\big)_N\big| \leqslant \frac{1}{2}\big(\big\|\widetilde{f}(t)\big\|_N^2 + \big\|\widetilde{u}_N(t)\big\|^2\big). \tag{5.24}$$

Moreover, by (2.25) and (3.1),

$$\big|\big(\widetilde{u}_N^2(t), \widetilde{u}_N(t)\big)_N\big| \leqslant \big\|\widetilde{u}_N(t)\big\|_{L^\infty}\big\|\widetilde{u}_N(t)\big\|^2 \leqslant cN^{1/2}\big\|\widetilde{u}_N(t)\big\|_{\chi^{(-1/2,-1/2)}}\big\|\widetilde{u}_N(t)\big\|^2. \tag{5.25}$$

On the other hand, we get from (3.8) with (3.9), lemma 3.5 and remark 3.1 that for any $v \in {}_0H^1_{\alpha,0,0,0}(\Lambda)$ and $\alpha \leqslant \frac{3}{2}$,

$$\begin{aligned} \|v\|_{\chi^{(-1/2,-1/2)}} &\leqslant \big\|v - v(0)\big\|_{\chi^{(-1/2,-1/2)}} + c\big|v(0)\big| \\ &\leqslant \big|v - v(0)\big|_{1,\chi^{(3/2,3/2)}} + c \max_{x\in[-1,1/2]}\big|v(x)\big| \\ &\leqslant c\big(|v|_{1,\chi^{(\alpha,0)}} + \|v\|_{1,\chi^{(\alpha,0)}}\big) \leqslant c|v|_{1,\chi^{(\alpha,0)}}. \end{aligned}$$

Thus for any $\varepsilon > 0$,

$$\big|\big(\widetilde{u}_N^2(t), \widetilde{u}_N(t)\big)_N\big| \leqslant \varepsilon\big\|\widetilde{u}_N(t)\big\|^2_{1,\alpha,0,0,0} + \frac{cN}{4\varepsilon}\big\|\widetilde{u}_N(t)\big\|^4. \tag{5.26}$$

Next, let $0 < q < 2$ and $\varepsilon = 2 - q$. Substituting (5.23), (5.24) and (5.26) into (5.22), we obtain that

$$\partial_t\big\|\widetilde{u}_N(t)\big\|^2 + q\big\|\widetilde{u}_N(t)\big\|^2_{1,\alpha,0,0,0} \leqslant c\big(1 + |||u_N|||_\infty\big)\big\|\widetilde{u}_N(t)\big\|^2 + cN\big\|\widetilde{u}_N(t)\big\|^4 + \big\|\widetilde{f}\big\|_N^2. \tag{5.27}$$

For description of the errors, let

$$E(v, t) = \big\|v(t)\big\|^2 + q\int_0^t \big\|v(s)\big\|^2_{1,\alpha,0,0,0}\,\mathrm{d}s$$

and

$$\rho(v, w, t) = \|v\|^2 + \int_0^t \|w(s)\|_N^2 \, \mathrm{d}s.$$

Integrating (5.27) with respect to t, we get that

$$E(\widetilde{u}_N, t) \leqslant \rho(\widetilde{u}_{N,0}, \widetilde{f}, t) + d(u_N) \int_0^t \big(E(\widetilde{u}_N, s) + NE^2(\widetilde{u}_N, s)\big) \, \mathrm{d}s$$

where $d(u_N)$ is a positive constant depending only on $\||u_N\||_\infty$. Finally, we use lemma 3.2 of Guo [18] to conclude that

Theorem 5.2. Let u_N be the solution of (5.20), and $\widetilde{u}_N$ be its error induced by $\widetilde{u}_{N,0}$ and $\widetilde{f}$. If there exist positive constants b_1 and b_2 depending only on $\||u_N\||_\infty$, such that for certain $t_1 \leqslant T$, $\rho(\widetilde{u}_{N,0}, \widetilde{f}, t_1) \leqslant b_1/N$, then for all $0 \leqslant t \leqslant t_1$,

$$E_N(\widetilde{u}_N, t) \leqslant \rho\big(\widetilde{u}_{N,0}, \widetilde{f}, t_1\big)\mathrm{e}^{b_2 t}. \tag{5.28}$$

We next deal with the convergence of scheme (5.20). Putting $U_N = \widehat{P}^1_{N,\alpha,0,0,0}U$, we get from (5.19) that

$$\begin{aligned}
&\big(\partial_t U_N(t), \phi\big)_N + a_{\alpha,N}\big(U_N(t), \phi\big) \\
&\quad = \big(2U_N(t) - U_N^2(t), \phi\big)_N + \big(f(t), \phi\big)_N + \sum_{j=1}^{5} G_j(t, \phi), \\
&\quad \forall \phi \in {}_0\mathcal{P}_N, \ t \in (0, T],
\end{aligned} \tag{5.29}$$

where

$$\begin{aligned}
G_1(t, \phi) &= \big(\partial_t U_N(t), \phi\big)_N - \big(\partial_t U(t), \phi\big), \\
G_2(t, \phi) &= \big(U(t), \phi\big) - \big(U_N(t), \phi\big)_N, \\
G_3(t, \phi) &= \big(\partial_x U_N(t), \partial_x \phi\big)_{\chi^{(\alpha,0)},N,*} - \big(\partial_x U(t), \partial_x \phi\big)_{\chi^{(\alpha,0)}}, \\
G_4(t, \phi) &= \big(U_N^2(t), \phi\big)_N - \big(U^2(t), \phi\big), \\
G_5(t, \phi) &= \big(f(t), \phi\big) - \big(f(t), \phi\big)_N.
\end{aligned}$$

Let $\widetilde{U}_N = u_N - U_N$. By subtracting (5.29) from (5.20), we obtain that for any $\phi \in {}_0\mathcal{P}_N$ and $t \in (0, T]$,

$$\big(\partial_t \widetilde{U}_N(t), \phi\big)_N + a_{\alpha,N}\big(\widetilde{U}_N(t), \phi\big) + \sum_{j=1}^{5} G_j(t, \phi) = \big(2\widetilde{U}_N(t) - 2U_N(t)\widetilde{U}_N(t) - \widetilde{U}_N^2(t), \phi\big)_N. \tag{5.30}$$

In addition,

$$\widetilde{U}_N(0) = \mathcal{R}_N U_0 - \widehat{P}^1_{N,\alpha,0,0,0} U_0.$$

Taking $\phi = \widetilde{U}_N(t)$ in (5.30) and comparing (5.30) with (5.21), we can derive an estimate like (5.27). But $u_N, \widetilde{u}_N$ and $|||u_N|||_\infty$ are now replaced by $U_N, \widetilde{U}_N$ and $|||U_N|||_\infty$, respectively. Thus, it remains to estimate $|G_j(t, \widetilde{U}_N)|$. We first use (3.27) to get that for $r \geqslant 1$,

$$\begin{aligned}\bigl|G_1\bigl(t, \widetilde{U}_N\bigr)\bigr| &\leqslant \bigl\|\partial_t\bigl(\widehat{P}^1_{N,\alpha,0,0,0}U(t) - U(t)\bigr)\bigr\|\,\bigl\|\widetilde{U}_N(t)\bigr\| \\ &\leqslant cN^{-2r}\bigl\|\partial_t U(t)\bigr\|^2_{r,\chi^{(\alpha,0)},*} + \tfrac{1}{2}\bigl\|\widetilde{U}_N(t)\bigr\|^2,\end{aligned}$$

$$\bigl|G_2\bigl(t, \widetilde{U}_N\bigr)\bigr| \leqslant cN^{-2r}\bigl\|U(t)\bigr\|^2_{r,\chi^{(\alpha,0)},*} + \tfrac{1}{2}\bigl\|\widetilde{U}_N(t)\bigr\|^2$$

and

$$\bigl|G_3\bigl(t, \widetilde{U}_N\bigr)\bigr| \leqslant \frac{cN^{-2r}}{4\varepsilon}\bigl\|U(t)\bigr\|^2_{r+1,\chi^{(\alpha,0)},*} + \varepsilon\bigl\|\widetilde{U}_N(t)\bigr\|^2_{1,\alpha,0,0,0}.$$

Next, by theorem 4.7, lemma 3.12 and the imbedding theorem, we get that for $r \geqslant 1,\ d > 1$ and $d' > 1/2$,

$$\begin{aligned}\bigl|G_4\bigl(t, \widetilde{U}_N\bigr)\bigr| &\leqslant \bigl|\bigl(\mathcal{R}_N\bigl(U_N^2(t) - U^2(t)\bigr), \widetilde{U}_N\bigr)\bigr| + \bigl|\bigl(\mathcal{R}_N U^2(t) - U^2(t), \widetilde{U}_N\bigr)\bigr| \\ &\leqslant \bigl\|U_N(t) + U(t)\bigr\|_\infty \bigl\|\mathcal{R}_N U(t) - U_N(t)\bigr\|\,\bigl\|\widetilde{U}_N(t)\bigr\| \\ &\quad + \bigl\|\mathcal{R}_N U^2(t) - U^2(t)\bigr\|\,\bigl\|\widetilde{U}_N\bigr\| \\ &\leqslant M(U)N^{-2r}\bigl(\bigl\|U(t)\bigr\|^2_{r,\chi^{(0,0)},*} + \bigl\|U(t)\bigr\|^2_{r,\chi^{(\alpha,0)},*} \\ &\quad + \bigl\|U(t)\bigr\|^2_{W^{[r/2],\infty}}\bigl\|U(t)\bigr\|^2_{r,\chi^{(0,0)},*}\bigr) + \bigl\|\widetilde{U}_N(t)\bigr\|^2 \\ &\leqslant M(U)N^{-2r}\bigl(\bigl\|U(t)\bigr\|^2_{r,\chi^{(0,0)},*} + \bigl\|U(t)\bigr\|^2_{r,\chi^{(\alpha,0)},*} \\ &\quad + \bigl\|U(t)\bigr\|^2_{r/2+d'}\bigl\|U(t)\bigr\|^2_{r,\chi^{(0,0)},*}\bigr) + \bigl\|\widetilde{U}_N(t)\bigr\|^2,\end{aligned}$$

where

$$M(U) = c\max_{0\leqslant t\leqslant T}\bigl(\bigl\|U(t)\bigr\|^2_d + \bigl\|U(t)\bigr\|^2_{d,\chi^{(\alpha,0)},*} + \bigl\|U(t)\bigr\|^2_{1,\alpha,0,0,0} + 1\bigr).$$

Moreover by theorem 4.7, we obtain that for $s \geqslant 1$,

$$\bigl|G_5\bigl(t, \widetilde{U}_N\bigr)\bigr| \leqslant cN^{-2s}\bigl\|f(t)\bigr\|^2_{s,\chi^{(0,0)},*} + \tfrac{1}{2}\bigl\|\widetilde{U}_N(t)\bigr\|^2.$$

Using theorem 4.7 and lemma 3.12 again, we get that for $s' \geqslant 1$,

$$\begin{aligned}\bigl\|\widetilde{U}_N(0)\bigr\|^2 &= \bigl\|\mathcal{R}_N U_0 - \widehat{P}^1_{N,\alpha,0,0,0}U_0\bigr\|^2 \leqslant 2\|\mathcal{R}_N U_0 - U_0\|^2 + 2\bigl\|\widehat{P}^1_{N,\alpha,0,0,0}U_0 - U_0\bigr\|^2 \\ &\leqslant cN^{-2s'}\bigl(\|U_0\|^2_{s',\chi^{(0,0)},*} + \|U_0\|^2_{s',\chi^{(\alpha,0)},*}\bigr).\end{aligned}$$

Finally, we reach the following result.

Theorem 5.3. Let U and u_N be the solutions of (5.18) and (5.20), respectively. Assume that

(i) $U \in L^2(0,T; H^{r+1}_{\chi^{(\alpha,0)},*}(\Lambda) \cap H^r_{\chi^{(0,0)},*}(\Lambda)) \cap L^\infty(0,T; {}_0H^1_{\alpha,0,0,0}(\Lambda) \cap H^d_{\chi^{(\alpha,0)},*}(\Lambda) \cap H^d(\Lambda) \cap H^{r/2+d'}(\Lambda)) \cap H^1(0,T; H^r_{\chi^{(\alpha,0)},*}(\Lambda))$, $r \geqslant 1$, $d > 1$, $d' > 1/2$,

(ii) $U_0 \in H^{s'}_{\chi^{(\alpha,0)},*}(\Lambda) \cap H^{s'}_{\chi^{(0,0)},*}(\Lambda)$, $f \in L^2(0,T; H^s_{\chi^{(0,0)},*}(\Lambda))$, $s, s' \geqslant 1$.

Then for all $0 \leqslant t \leqslant T$,

$$E_N(u_N - U_N, t) \leqslant M^*\bigl(N^{-2r} + N^{-2s} + N^{-2s'}\bigr), \tag{5.31}$$

where M^* is a positive constant depending only on the norms of U, U_0 and f in the mentioned spaces.

Remark 5.2. The condition on U in theorem 5.3 can be replaced by

$$U \in L^2\bigl(0,T; H^{r+1}_{\chi^{(\alpha,0)},*}(\Lambda)\bigr) \cap L^4\bigl(0,T; H^r_{\chi^{(0,0)},*}(\Lambda) \cap W^{[r/2],\infty}(\Lambda)\bigr)$$
$$\cap L^\infty\bigl(0,T; {}_0H^1_{\alpha,0,0,0}(\Lambda) \cap H^d_{\chi^{(\alpha,0)},*}(\Lambda) \cap H^d(\Lambda)\bigr) \cap H^1\bigl(0,T; H^r_{\chi^{(\alpha,0)},*}(\Lambda)\bigr).$$

Remark 5.3. The Jacobi pseudospectral method is also applicable to differential equations on infinite intervals. For example, we consider the logistic equation governing the population of budworms in an unbounded forest, say $\Lambda^* = \{y \mid 0 < y < \infty\}$. Suppose that the boundary condition at $y = 0$ is lethal, and the population $V(y,t)$ grows infinitely as $y \to \infty$, but at least $\mathrm{e}^{-y}\partial_y V(y,t) \to 0$. This problem is of the form

$$\begin{cases} \partial_t V(y,t) - \partial_y^2 V(y,t) = V(y,t)\bigl(1 - V(y,t)\bigr), & y \in \Lambda^*, 0 < t \leqslant T, \\ V(0,t) = \lim\limits_{y\to\infty} \mathrm{e}^{(-1/4)y}\partial_y V(y,t) = 0, & 0 \leqslant t \leqslant T, \\ V(y,0) = V_0(y), & y \in \Lambda^*. \end{cases} \tag{5.32}$$

Now we make the variable transformation (see Guo [20]),

$$y(x) = -2\ln(1-x) + 2\ln 2$$

and set $U(x,t) = V(y(x),t)$ and $U_0(x) = V_0(y(x))$. Then the original problem becomes

$$\begin{cases} \partial_t U(x,t) - \frac{1}{4}(1-x)\partial_x\bigl((1-x)\partial_x U(x,t)\bigr) = U(x,t)\bigl(1 - U(x,t)\bigr), \\ \quad x \in \Lambda,\ 0 < t \leqslant T, \\ U(-1,t) = \lim\limits_{x\to 1}(1-x)^{3/2}\partial_x U(x,t) = 0, \quad 0 \leqslant t \leqslant T, \\ U(x,0) = U_0(x). \end{cases} \tag{5.33}$$

So we can construct the corresponding pseudospectral scheme to resolve (5.33) numerically, and analyze its stability and convergence.

6. Numerical results

This section is for some numerical results. We first consider problem (5.1) with $a(x) = 1 - x^2$ and $c(x) \equiv 1$. The test function is

$$U(x) = x \arcsin x + \sqrt{1 - x^2} - 1. \tag{6.1}$$

It can be checked that $|\partial_x^2 U(x)| \to \infty$ as $|x| \to 1$, but $U \in H^r_{\chi^{(1,1)},*}(\Lambda)$, $3 < r < 4$. We solve it by (5.6) with $\alpha = \beta = 1$ and $\gamma = \delta = 0$. By theorem 5.1, scheme (5.6) is convergent in this case. Let $u_N(x)$ be the numerical solution. For description of numerical errors, let

$$E_1(v) = \left(\sum_{j=0}^{N} \big(U\big(\zeta_{G,N,j}^{(0,0)}\big) - v\big(\zeta_{G,N,j}^{(0,0)}\big)\big)^2 \omega_{G,N,j}^{(0,0)} \right)^{1/2}$$

and

$$E_2(v) = \left(\frac{\sum_{j=0}^{N} (U(\zeta_{G,N,j}^{(0,0)}) - v(\zeta_{G,N,j}^{(0,0)}))^2 \omega_{G,N,j}^{(0,0)}}{\sum_{j=0}^{N} |U(\zeta_{G,N,j}^{(0,0)})|^2 \omega_{G,N,j}^{(0,0)}} \right)^{1/2}.$$

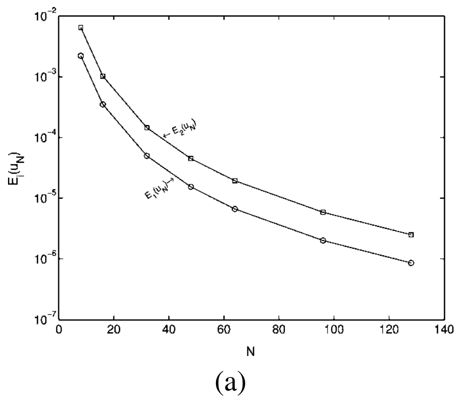
The errors $E_1(u_N)$ and $E_2(u_N)$ with different N are illustrated in figure 1(a).

We also consider (5.1) with $a(x) = 1 - x^2$ and $c(x) \equiv 1$. The test function is

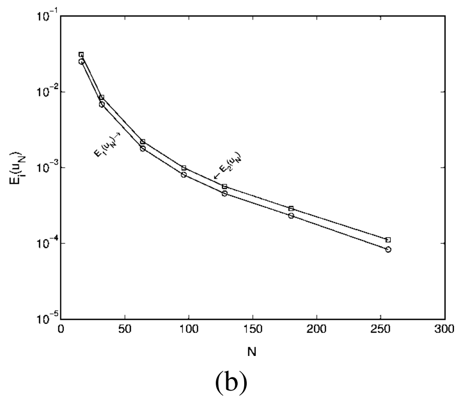
$$U(x) = \big(1 - x^2\big) \log_{10}\big(1 - x^2\big) + 1. \tag{6.2}$$

Clearly, $\partial_x U$ has a logarithmic singularity. We also use scheme (5.6) with $\alpha = \beta = 1$ and $\gamma = \delta = 0$ to solve this problem numerically. Let $u_N(x)$ be the numerical solution. The errors $E_1(u_N)$ and $E_2(u_N)$ are presented in figure 1(b).

Figure 1 shows that scheme (5.6) provides very accurate numerical solution even for small N. It also indicates that the numerical solution converges fast as N increases, and that scheme (5.6) possesses the spectral accuracy. It coincides with the theoretical analysis in section 5.

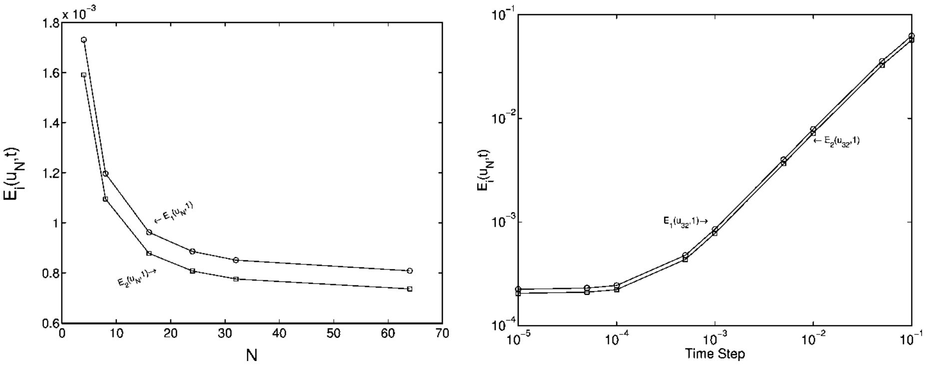


(a)



(b)

Figure 1. The errors $E_1(u_N)$ and $E_2(u_N)$.



(a) (b)

Figure 2. The errors $E_1(u_N, t)$ and $E_2(u_N, t)$: (a) $\tau = 10^{-3},\ t = 1$; (b) $N = 32,\ t = 1$.

Next, we consider problem (5.18), and take the test function

$$U(x, t) = (1 - x)^{\eta} \sin\left(\frac{1}{2}(1 + x)(1 + t)\right). \tag{6.3}$$

Clearly, the regularity of $U(x, t)$ depends on η essentially. We use scheme (5.20) to solve (5.18). In actual computation, we advance in time by using Runge–Kutta method of fourth order with mesh size τ. Let $u_N(x, t)$ be the numerical solution. For description of numerical errors, let

$$E_1(u_N, t) = \left(\sum_{j=0}^{N} \left(U\left(\zeta_{R,N,j}^{(0,0)}, t\right) - u_N\left(\zeta_{R,N,j}^{(0,0)}, t\right)\right)^2 \omega_{R,N,j}^{(0,0)}\right)^{1/2}$$

and

$$E_2(u_N, t) = \left(\frac{\sum_{j=0}^{N}(U(\zeta_{R,N,j}^{(0,0)}, t) - u_N(\zeta_{R,N,j}^{(0,0)}, t))^2 \omega_{R,N,j}^{(0,0)}}{\sum_{j=0}^{N} U^2(\zeta_{R,N,j}^{(0,0)}, t)\omega_{R,N,j}^{(0,0)}}\right)^{1/2}.$$

Let $\alpha = 1$ and $\eta = -0.01$. The numerical errors at $t = 1$ of scheme (5.20) with $\tau = 10^{-3}$ and different N are illustrated in figure 2(a). While the errors at $t = 1$ with $N = 32$ and different time step τ are given in figure 2(b). Figure 2 indicates that the numerical solution converges as N increases and τ decreases. The errors $E_1(u_N, t)$ and $E_2(u_N, t)$ at different time t with $N = 32$ and $\tau = 10^{-3}$ are presented in figure 3. It shows the stability of computaions. Moreover, we illustrate the exact solution (6.3) at $t = 1$ with $\eta = -0.2$ and the corresponding numerical solution of scheme (5.20) with $N = 32$ and $\tau = 10^{-3}$ in figure 4. We find that the numerical solution fits the singularity of the exact solution very well.

Remark 6.1. In this paper, we only considered the Jacobi interpolation approximations in one dimension. It is more interesting to develop the Jacobi interpolation in multiple

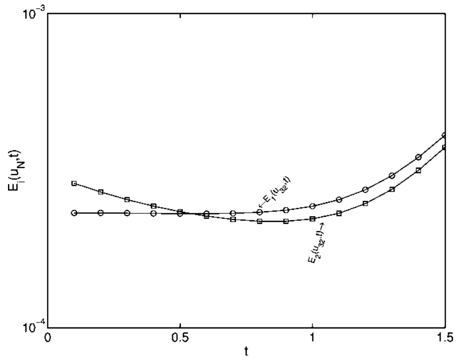


Figure 3. The errors with $N = 32,\ \tau = 10^{-3}$.

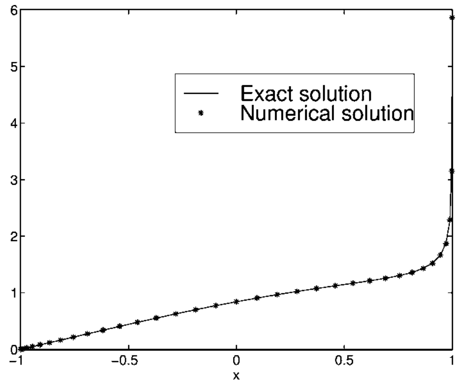


Figure 4. $u_{32}(x, 1)$ fits $U(x, 1)$.

dimensions and their applications to numerical solutions of differential equations. Some results have been obtained, see Wang and Guo [42], and Guo and Wang [23].

Remark 6.2. The Jacobi interpolation approximations are also widely used in numerical solutions of singular integral equations, see Junghanns [28], Junghanns and Silbermann [29], Karpenko [30], Elliott [14], Chawia and Ramakrishnan [10], Krenk [32], Ioakimidis and Theocrais [25,26].

Remark 6.3. In this paper, we considered the Jacobi interpolation approximations with $\alpha, \beta > -1$. But the Jacobi interpolation approximations with $\alpha, \beta \leqslant -1$ are used for supersingular boundary integral equations arising in boundary element methods for differential equations, see Stephan and Suri [37], and Tran and Stephan [40,41].

Appendix A. The proof of (2.7)

In fact, $\int J_l^{(\alpha,\beta)}(x)\,\mathrm{d}x \in \mathcal{P}_{l+1}$. So it can be expressed as

$$\int J_l^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{k=0}^{l+1} d_k J_k^{(\alpha,\beta)}(x). \tag{A.1}$$

By differentiating (A.1) and using (2.10), we deduce that

$$d_k = \frac{1}{\gamma_k^{(\alpha,\beta)}\lambda_k^{(\alpha,\beta)}} \int_\Lambda J_l^{(\alpha,\beta)}(x)\partial_x J_k^{(\alpha,\beta)}(x)\chi^{(\alpha+1,\beta+1)}(x)\,\mathrm{d}x, \quad 1 \leqslant k \leqslant l+1. \tag{A.2}$$

Since $J_l^{(\alpha,\beta)}(x)$ is orthogonal to any polynomial in $\mathcal{P}_{l-1}$, with respect to the inner product of $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$, we have from (A.2) that $d_k = 0$, for $k \leqslant l-2$. For clarity, we denote the expression (A.1) by

$$J_l^{(\alpha,\beta)}(x) = a_l\partial_x J_{l+1}^{(\alpha,\beta)}(x) + b_l\partial_x J_l^{(\alpha,\beta)}(x) + c_l\partial_x J_{l-1}^{(\alpha,\beta)}(x). \tag{A.3}$$

We first compare the coefficients of the term x^m at the both sides of (A.3). We obtain from (2.2) and (2.6) that

$$a_l = \frac{2K_l^{(\alpha,\beta)}}{(l+\alpha+\beta+2)K_l^{(\alpha+1,\beta+1)}} = \frac{2(l+\alpha+\beta+1)}{(2l+\alpha+\beta+1)(2l+\alpha+\beta+2)}.$$

Letting $x = \pm 1$ in (A.3) and using (2.3), we get a system of equations for the unknown coefficients b_l and c_l. By solving it, we obtain that

$$\begin{aligned} b_l &= \frac{2(\alpha-\beta)}{(2l+\alpha+\beta)(2l+\alpha+\beta+2)}, \\ c_l &= \frac{-2(l+\alpha)(l+\beta)}{(l+\alpha+\beta)(2l+\alpha+\beta)(2l+\alpha+\beta+1)}. \end{aligned}$$

Finally, by (A.3),

$$\begin{aligned} \int_{-1}^{x} J_l^{(\alpha,\beta)}(y)\,\mathrm{d}y = {} & a_l\big(J_{l+1}^{(\alpha,\beta)}(x) - J_{l+1}^{(\alpha,\beta)}(-1)\big) + b_l\big(J_l^{(\alpha,\beta)}(x) - J_l^{(\alpha,\beta)}(-1)\big) \\ & + c_l\big(J_{l-1}^{(\alpha,\beta)}(x) - J_{l-1}^{(\alpha,\beta)}(-1)\big) \end{aligned}$$

which completes the proof of (2.7).

Appendix B. The proof of (2.26)

Let

$$\psi(x) = \big(J_N^{(\alpha,\beta)}(x)\big)^2 + \frac{1}{N^2}\big(1-x^2\big)\big(\partial_x J_N^{(\alpha,\beta)}(x)\big)^2.$$

Due to (2.2) and (2.6), $\psi(x) \in \mathcal{P}_{2N-1}$. So by (2.25),

$$\begin{aligned} \big(J_N^{(\alpha,\beta)}, J_N^{(\alpha,\beta)}\big)_{\chi^{(\alpha,\beta)},L,N} &= (1,\psi)_{\chi^{(\alpha,\beta)},L,N} = \int_\Lambda \psi(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x \\ &= \big(J_N^{(\alpha,\beta)}, J_N^{(\alpha,\beta)}\big)_{\chi^{(\alpha,\beta)}} + \frac{1}{N^2}\big(\partial_x J_N^{(\alpha,\beta)}, \partial_x J_N^{(\alpha,\beta)}\big)_{\chi^{(\alpha+1,\beta+1)}}. \end{aligned}$$

We deduce from (2.8) and (2.10) that

$$\big(J_N^{(\alpha,\beta)}, J_N^{(\alpha,\beta)}\big)_{\chi^{(\alpha,\beta)},L,N} = \left(2 + \frac{\alpha+\beta+1}{N}\right)\gamma_N^{(\alpha,\beta)}. \tag{B.1}$$

For any $\phi \in \mathcal{P}_N$, let

$$\phi(x) = \sum_{j=0}^{N} \widehat{\phi}_j J_j^{(\alpha,\beta)}(x).$$

By (2.25) and (B.1),

$$\|\phi\|^2_{\chi^{(\alpha,\beta)},L,N} = \sum_{j=0}^{N-1} \widehat{\phi}_j^2 \gamma_j^{(\alpha,\beta)} + \left(2 + \frac{\alpha+\beta+1}{N}\right) \widehat{\phi}_N^2 \gamma_N^{(\alpha,\beta)}.$$

Thus

$$\sum_{j=0}^{N} \widehat{\phi}_j^2 \gamma_j^{(\alpha,\beta)} \leqslant \|\phi\|^2_{\chi^{(\alpha,\beta)},L,N} \leqslant \left(2 + \frac{\alpha+\beta+1}{N}\right) \sum_{j=0}^{N} \widehat{\phi}_j^2 \gamma_j^{(\alpha,\beta)}$$

which leads to (2.26).

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