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Stair Laguerre pseudospectral method for differential equations on the half line

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Dedicated to Charles A. Micchelli on the occasion of his 60th birthday

A stair Laguerre pseudospectral method is proposed for numerical solutions of differential equations on the half line. Some approximation results are established. A stair Laguerre pseudospcetral scheme is constructed for a model problem. The convergence is proved. The numerical results show that this new method provides much more accurate numerical results than the standard Laguerre spectral method.

Keywords: stair Laguerre pseudospectral method, half line.

Mathematics subject classifications (2000): 65N35, 41A10.

1. Introduction

Spectral method employs global orthogonal polynomials as basis in spatial discretization of differential equations. It benefits from the rapid convergence of orthogonal systems, and so often provides good numerical results, see, e.g., [2, 3, 6, 7]. The usual spectral method is available for bounded domains. But it is also interesting and challenging to consider spectral method for unbounded domains. For instance, Funaro [5], Guo [8], and Guo and Xu [11] developed the Hermite spectral and pseudospectral methods for the whole line. While Maday et al. [13], Coulaud et al. [4], Irazo and Falqués [12], Guo and Shen [10], Xu and Guo [18], and Guo and Ma [9] proposed various Laguerre spectral and pseudospcetral methods for the half line. Theoretically, the

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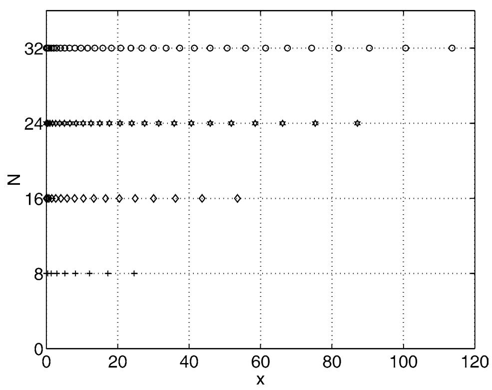

Figure 1. Distribution of interpolation nodes.

larger the number of terms involved in the Laguerre expansion of numerical solutions, the smaller the errors of numerical solutions. However, there are still two difficulties in actual computation, if we use the standard Laguerre spectral and pseudospectral methods.

The first difficulty comes from the appearance of the weight function $\omega(x) = \mathrm{e}^{-x}$. Indeed, many problems are well-posed in non-weighted Sobolev spaces, but not well-posed in the weighted Sobolev spaces. On the other hand, the errors of the Laguerre spectral method are measured in the weighted Sobolev spaces. Thus for large x, the numerical errors might be big, even the errors in the weighted norms are small. To remedy this deficiency, Guo and Shen [10] reformed differential equations on the half line by using certain variable transformation so that the resulting problems are well-posed in the weighted space, and then solved numerically by the Laguerre approximation. In this case, the numerical errors of the original problems are measured in the non-weighted spaces. Shen [15] also considered a modified Laguerre approximation and obtained similar results for the solutions decaying fast at the infinity.

The second difficulty is caused by the distribution of the Laguerre–Gauss–Radau interpolation nodes. Let N be the number of interpolation nodes. As shown in figure 1, the distances between adjacent nodes increase very fast as N increases, especially for those far from the left endpoint. Therefore, the numerical results only can fit the exact solutions roughly for large x.

The aim of this paper is to develop a stair Laguerre pseudospectral method for differential equations on the half line. Firstly, we follow the idea of Guo and Shen [10] to derive some alternative formulations of the original problems, which are well-posed in the weighted Sobolev spaces. Next, we use the standard Laguerre pseudospectral method with relatively small mode N_0 (the number of interpolation nodes) to obtain the first numerical solution $u^0_{N_0}(x)$ on the half line. Since the largest interpolation node $x^0_{N_0} = \mathrm{O}(\sqrt{N_0})$ is not big, the numerical solution fits the exact solution well at all nodes $x^0_j \leqslant x^0_{M_0}$, $M_0 \leqslant N_0$. Furthermore, we use the shift Laguerre pseudospectral method with N_1 interpolation nodes for the same problem on the infinite subinterval $[x^0_{M_0}, \infty)$, and obtain the second numerical solution at the nodes $x^1_j \leqslant x^1_{M_1}$, $M_1 \leqslant N_1$. By repeating the above procedure, we get the numerical solutions at the nodes $x^0_0, \dots, x^0_{M_0}, x^1_0, \dots, x^1_{M_1}, \dots$. Since all N_k are not big, the above nodes are located densely and almost uniformly. Therefore, the numerical solutions fit the exact solutions properly even for large x. The small N_k also avoid bad condition numbers of the matrices in the corresponding discrete systems with big N_k, as required in the usual Laguerre pseudospectral method. Moreover, we do not need to resolve large systems and so saves a lot of work. Furthermore, we can use this trick locally to improve the accuracy of numerical results on any subintervals where the exact solutions change rapidly.

This paper is organized as follows. In the next section, some basic results on the stair Laguerre approximations are established, which form the mathematical foundation of the related algorithms. In section 3, a stair Laguerre pseudospectral scheme is proposed for a model problem on the half line. Its convergence is proved. In section 4, we describe numerical implementations. A reasonable choice of base functions simplifies calculation and saves work. Some numerical results are presented. They demonstrate that this new approach provides much more accurate numerical solution than the standard Laguerre spectral method. The final section is for some concluding remarks.

2. Some basic results

Let $I = (a, b)$, $0 \leqslant a < b \leqslant \infty$, and $\chi(x)$ be a certain weight function on I in the usual sense. Define

$$L^2_\chi(I) = \{v \mid v \text{ is measurable on } I \text{ and } \|v\|_{\chi,I} < \infty\},$$

equipped with the following inner product and norm

$$(u, v)_{\chi,I} = \int_I u(x)v(x)\chi(x)\,\mathrm{d}x, \qquad \|v\|_{\chi,I} = (v, v)^{1/2}_{\chi,I}.$$

For simplicity, let $\partial_x v(x) = (\partial/\partial x)v(x)$, etc. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $m \in \mathbb{N}$,

$$H^m_\chi(I) = \{v \mid \partial^k_x v \in L^2_\chi(I),\ 0 \leqslant k \leqslant m\},$$

with the semi-norm and the norm

$$|v|_{m,\chi,I} = \left\|\partial^m_x v\right\|_{\chi,I}, \qquad \|v\|_{m,\chi,I} = \left(\sum_{k=0}^{m} |v|^2_{k,\chi,I}\right)^{1/2}.$$

For any real $r > 0$, we define the space $H^r_\chi(I)$ by space interpolation. Its semi-norm and norm are denoted by $|\cdot|_{r,\chi,I}$ and $\|\cdot\|_{r,\chi,I}$, respectively. The space $H^r_{0,\chi}(I)$ stands for the closure of the set $\mathcal{D}(I)$ consisting of all infinitely differential functions with compact support in I. For $\chi(x) \equiv 1$, we drop the subscript χ in the notations for simplicity.

Now, let $\Lambda = (0, \infty)$, $\omega(x) = \mathrm{e}^{-x}$, and $\mathcal{L}_l(x)$ be the Laguerre polynomial of degree l, defined by

$$\mathcal{L}_l(x) = \frac{1}{l!}\mathrm{e}^x \partial_x^l\big(x^l \mathrm{e}^{-x}\big), \quad l = 0, 1, \ldots.$$

They satisfy the equations

$$\partial_x\big(x\mathrm{e}^{-x}\partial_x\mathcal{L}_l(x)\big) + l\mathrm{e}^{-x}\mathcal{L}_l(x) = 0, \quad x \in \Lambda, \tag{2.1}$$

and

$$\mathcal{L}_l(x) = \partial_x\mathcal{L}_l(x) - \partial_x\mathcal{L}_{l+1}(x), \quad l \geqslant 0. \tag{2.2}$$

The set of Laguerre polynomials is the $L^2_\omega(\Lambda)$-orthogonal system, namely,

$$\int_\Lambda \mathcal{L}_l(x)\mathcal{L}_m(x)\omega(x)\,\mathrm{d}x = \delta_{l,m} \tag{2.3}$$

where $\delta_{l,m}$ is the Kronecher function. Moreover, by (2.1) and (2.3),

$$\int_\Lambda \partial_x\mathcal{L}_l(x)\partial_x\mathcal{L}_m(x)x\omega(x)\,\mathrm{d}x = l\delta_{l,m}. \tag{2.4}$$

Next, let $0 = a_0 < a_1 < \cdots < a_K < \infty$, $\Lambda_k = (a_k, \infty)$, and $\omega_k(x) = \omega(x-a_k) = \mathrm{e}^{a_k-x}$. The index set $\mathbb{K} = \{k \mid 0 \leqslant k \leqslant K\}$. For any $k \in \mathbb{K}$, define

$${}_0H^1_{\omega_k}(\Lambda_k) = \big\{v \mid v \in H^1_{\omega_k}(\Lambda_k),\ v(a_k) = 0\big\}.$$

In numerical analysis, we need the following imbedding inequalities.

Lemma 2.1. For any $v \in H^1_{\omega_k}(\Lambda_k)$ and $k \in \mathbb{K}$,

$$\sup_{x\in\bar\Lambda_k}\big|v(x)\omega_k^{1/2}(x)\big| \leqslant \|v\|_{\omega_k,\Lambda_k} + \sqrt{2}\|v\|^{1/2}_{\omega_k,\Lambda_k}|v|^{1/2}_{1,\omega_k,\Lambda_k} \leqslant \sqrt{2}\|v\|_{1,\omega_k,\Lambda_k}. \tag{2.5}$$

Moreover, for any $v \in {}_0H^1_{\omega_k}(\Lambda_k)$,

$$\|v\|_{\omega_k,\Lambda_k} \leqslant 2|v|_{1,\omega_k,\Lambda_k}. \tag{2.6}$$

Proof. For any $x \in \bar\Lambda_k$, we get from the Cauchy inequality that

$$\begin{aligned}
v^2(x)\omega_k(x) &= -\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}y}\big(v^2(y)\omega_k(y)\big)\,\mathrm{d}y \\
&= \int_x^\infty v^2(y)\omega_k(y)\,\mathrm{d}y - 2\int_x^\infty v(y)\partial_y v(y)\omega_k(y)\,\mathrm{d}y \\
&\leqslant \|v\|^2_{\omega_k,\Lambda_k} + 2\|v\|_{\omega_k,\Lambda_k}|v|_{1,\omega_k,\Lambda_k}.
\end{aligned}$$

This leads to (2.5). Next, for any $v \in {}_0H^1_{\omega_k}(\Lambda_k)$ and $x \in \bar{\Lambda}_k$,

$$v^2(x)\omega_k(x) = \int_{a_k}^{x} \frac{\mathrm{d}}{\mathrm{d}y}\bigl(v^2(y)\omega_k(y)\bigr)\,\mathrm{d}y$$
$$= -\int_{a_k}^{x} v^2(y)\omega_k(y)\,\mathrm{d}y + 2\int_{a_k}^{x} v(y)\partial_y v(y)\omega_k(y)\,\mathrm{d}y$$

whence

$$v^2(x)\omega_k(x) + \int_{a_k}^{x} v^2(y)\omega_k(y)\,\mathrm{d}y \leqslant 2\|v\|_{\omega_k,\Lambda_k}|v|_{1,\omega_k,\Lambda_k}.$$

Letting $x \to \infty$ in the above, we get (2.6). □

We next consider some orthogonal projections. Let $N = (N_0, N_1, \ldots, N_K) \in \mathbb{N}^{K+1}$, and $\mathcal{P}_{N_k}(\Lambda_k)$ be the set of all algebraic polynomials of degree at most N_k on Λ_k. Denote by c a generic positive constant independent of any function and N_k.

In order to describe the approximation error precisely, we introduce the weighted Sobolev space

$$A^r(\Lambda_k) = \bigl\{v \mid v \text{ is measurable on } \Lambda_k \text{ and } \|v\|_{A^r,\Lambda_k} < \infty\bigr\},$$

equipped with the semi-norm and norm

$$|v|_{A^r,\Lambda_k} = \bigl\|(x-a_k)^{r/2}\partial_x^r v\bigr\|_{\omega_k,\Lambda_k}, \qquad \|v\|_{A^r,\Lambda_k} = \left(\sum_{j=0}^{r} |v|^2_{A^j,\Lambda_k}\right)^{1/2}.$$

For real $r > 0$, we define the space $A^r(\Lambda_k)$ by space interpolation.

The $L^2_{\omega_k}(\Lambda_k)$-orthogonal projection $P_{N_k} : L^2_{\omega_k}(\Lambda_k) \to \mathcal{P}_{N_k}(\Lambda_k)$ is a mapping such that for any $v \in L^2_{\omega_k}(\Lambda_k)$,

$$(P_{N_k}v - v, \phi)_{\omega_k,\Lambda_k} = 0, \quad \forall\phi \in \mathcal{P}_{N_k}(\Lambda_k).$$

Lemma 2.2. For any $v \in A^r(\Lambda_k)$ and integer $r \geqslant 0$,

$$\|P_{N_k}v - v\|_{\omega_k,\Lambda_k} \leqslant cN_k^{-r/2}|v|_{A^r,\Lambda_k}, \quad k \in \mathbb{K}. \tag{2.7}$$

We can follow the same line as in the proof of Wang and Guo [17, theorem 4.1] to prove the above result, see appendix.

We next consider the $H^1_{\omega_k}(\Lambda_k)$-orthogonal projection. Let $\nu = (\nu_1, \nu_2)$, $\nu_1, \nu_2 > 0$, and

$$a^{\nu}_{\omega_k,\Lambda_k}(u, v) = \nu_1(\partial_x u, \partial_x v)_{\omega_k,\Lambda_k} + \nu_2(u, v)_{\omega_k,\Lambda_k}.$$

The orthogonal projection $P^1_{N_k} : H^1_{\omega_k}(\Lambda_k) \to \mathcal{P}_{N_k}(\Lambda_k)$ is a mapping such that for any $v \in H^1_{\omega_k}(\Lambda_k)$,

$$a^{\nu}_{\omega_k,\Lambda_k}\bigl(P^1_{N_k}v - v, \phi\bigr) = 0, \quad \forall\phi \in \mathcal{P}_{N_k}(\Lambda_k). \tag{2.8}$$

Lemma 2.3. If $\partial_x v \in A^{r-1}(\Lambda_k)$ and integer $r \geqslant 1$, then

$$\left\| P^1_{N_k} v - v \right\|_{1,\omega_k,\Lambda_k} \leqslant c N_k^{1/2-r/2} |\partial_x v|_{A^{r-1},\Lambda_k}, \quad k \in \mathbb{K}. \tag{2.9}$$

Proof. For any $\phi \in \mathcal{P}_{N_k}(\Lambda_k)$,

$$\begin{aligned} C_1(\nu) \left\| P^1_{N_k} v - v \right\|^2_{1,\omega_k,\Lambda_k} &\leqslant a^\nu_{\omega_k,\Lambda_k}\left(P^1_{N_k} v - v,\, P^1_{N_k} v - v\right) = a^\nu_{\omega_k,\Lambda_k}\left(P^1_{N_k} v - v, \phi - v\right) \\ &\leqslant C_2(\nu) \left\| P^1_{N_k} v - v \right\|_{1,\omega_k,\Lambda_k} \|\phi - v\|_{1,\omega_k,\Lambda_k}, \end{aligned}$$

where $C_1(\nu) = \min(\nu_1, \nu_2)$ and $C_2(\nu) = \max(\nu_1, \nu_2)$. Thus

$$\left\| P^1_{N_k} v - v \right\|_{1,\omega_k,\Lambda_k} \leqslant c \inf_{\phi \in \mathcal{P}_{N_k}(\Lambda_k)} \|\phi - v\|_{1,\omega_k,\Lambda_k}. \tag{2.10}$$

Take

$$\phi(x) = v(a_k) + \int_{a_k}^x P_{N_k-1} \partial_y v(y)\, \mathrm{d}y.$$

Clearly, $\phi - v \in {}_0H^1_{\omega_k}(\Lambda_k)$. By (2.6), (2.10) and lemma 2.2,

$$\begin{aligned} \left\| P^1_{N_k} v - v \right\|_{1,\omega_k,\Lambda_k} &\leqslant c\|\phi - v\|_{1,\omega_k,\Lambda_k} \leqslant c|\phi - v|_{1,\omega_k,\Lambda_k} = c\|P_{N_k-1}\partial_x v - \partial_x v\|_{\omega_k,\Lambda_k} \\ &\leqslant c N_k^{1/2-r/2} |\partial_x v|_{A^{r-1},\Lambda_k}. \end{aligned}$$

This completes the proof. □

We now turn to the Laguerre–Gauss–Radau interpolation approximation. For $M \in \mathbb{N}$, let $\{x_j\}_{j=0}^M$ be the set of the Laguerre–Gauss–Radau interpolation nodes, i.e., $x_0 = 0$, and $\{x_j\}_{j=1}^M$ be the zeros of the polynomial $\partial_x \mathcal{L}_{M+1}(x)$. The corresponding quadrature weights are

$$\omega_j = \frac{1}{(M+1)\mathcal{L}^2_M(x_j)}, \quad 0 \leqslant j \leqslant M.$$

We know from [16] that

$$\int_\Lambda \phi(x)\psi(x)\omega(x)\, \mathrm{d}x = \sum_{j=0}^M \phi(x_j)\psi(x_j)\omega_j, \quad \forall \phi \cdot \psi \in \mathcal{P}_{2M}(\Lambda). \tag{2.11}$$

Further, let

$$x^k_j = a_k + x_j, \qquad \omega^k_j = \omega_j, \quad 0 \leqslant j \leqslant N_k, \qquad \Lambda_{N_k} = \left\{x^k_j,\ 0 \leqslant j \leqslant N_k\right\}, \quad k \in \mathbb{K}.$$

For simplicity, we introduce the discrete inner product

$$(u, v)_{\omega_k,N_k,\Lambda_k} = \sum_{j=0}^{N_k} u\left(x^k_j\right) v\left(x^k_j\right) \omega^k_j, \quad k \in \mathbb{K}.$$

By (2.11),

$$(\phi, \psi)_{\omega_k, N_k, \Lambda_k} = (\phi, \psi)_{\omega_k, \Lambda_k}, \quad \forall \phi \cdot \psi \in \mathcal{P}_{2N_k}(\Lambda_k). \tag{2.12}$$

For any $v \in C(\bar{\Lambda}_k)$, the Laguerre–Gauss–Radau interpolant $\mathcal{I}_{N_k} v(x) \in \mathcal{P}_{N_k}(\Lambda_k)$, satisfying

$$\mathcal{I}_{N_k} v(x) = v(x), \quad x \in \Lambda_{N_k},\ k \in \mathbb{K}.$$

To describe the interpolation approximation result, we introduce the space

$$B^r(\Lambda_k) = \{ v \mid v \in A^r(\Lambda_k) \quad \text{and} \quad \partial_x v \in A^{r-1}(\Lambda_k) \}, \quad r \geqslant 1,$$

with the semi-norm

$$|v|_{B^r, \Lambda_k} = \left(|v|^2_{A^r, \Lambda_k} + |\partial_x v|^2_{A^{r-1}, \Lambda_k} \right)^{1/2}.$$

Lemma 2.4. Let r be an integer. For any $v \in B^r(\Lambda_k)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\| \mathcal{I}_{N_k} v - v \|_{\mu, \omega_k, \Lambda_k} \leqslant c (\ln N_k)^{1/2} N_k^{\mu + 1/2 - r/2} |v|_{B^r, \Lambda_k}, \quad k \in \mathbb{K}. \tag{2.13}$$

Proof. Let $\widehat{\mathcal{I}}_M$ be the standard Laguerre–Gauss–Radau interpolation operator associated with the interpolation nodes $\{x_j\}_{j=0}^M$ and weights $\{\omega_j\}_{j=0}^M$. By theorem 4.4 of Wang and Guo [17], we have that for any $u \in B^r(\Lambda)$ and $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$\left\| \widehat{\mathcal{I}}_M u - u \right\|_{\mu, \omega, \Lambda} \leqslant c (\ln M)^{1/2} M^{\mu + 1/2 - r/2} |u|_{B^r, \Lambda}. \tag{2.14}$$

Now, let

$$v(x) = \hat{v}(\hat{x}), \quad x = \hat{x} + a_k,\ x \in \Lambda_k,\ \hat{x} \in \Lambda.$$

By the definition of $\mathcal{I}_{N_k}$, we find that $\widehat{\mathcal{I}_{N_k} v} = \widehat{\mathcal{I}}_{N_k} \hat{v}$. This fact with (2.14) leads to that

$$\begin{aligned} \| \mathcal{I}_{N_k} v - v \|_{\mu, \omega_k, \Lambda_k} &= \left\| \widehat{\mathcal{I}_{N_k} v} - \hat{v} \right\|_{\mu, \omega, \Lambda} = \left\| \hat{\mathcal{I}}_{N_k} \hat{v} - \hat{v} \right\|_{\mu, \omega, \Lambda} \\ &\leqslant c (\ln N_k)^{1/2} N_k^{\mu + 1/2 - r/2} |\hat{v}|_{B^r, \Lambda} \leqslant c (\ln N_k)^{1/2} N_k^{\mu + 1/2 - r/2} |v|_{B^r, \Lambda_k}. \end{aligned}$$

This completes the proof. □

3. Stair Laguerre pseudospectral scheme

We consider the following model equation

$$\begin{cases} -\partial_x^2 V(x) + \lambda V(x) = F(x), & \lambda > 0,\ x \in \Lambda, \\ V(0) = V_0, & \lim\limits_{x \to \infty} V(x) = \lim\limits_{x \to \infty} \partial_x V(x) = 0. \end{cases} \tag{3.1}$$

In order to derive a well-posed weighted variational formulation, we follow the idea of Guo and Shen [10] to make the variable transformation

$$U(x) = \mathrm{e}^{x/2} V(x), \qquad f(x) = \mathrm{e}^{x/2} F(x). \tag{3.2}$$

By (3.2), problem (3.1) is changed into

$$\begin{cases} -\partial_x^2 U(x) + \partial_x U(x) + \left(\lambda - \dfrac{1}{4}\right) U(x) = f(x), & \lambda > 0,\ x \in \Lambda, \\ U(0) = U_0 = V_0, & \lim\limits_{x\to\infty} \mathrm{e}^{-x/2} U(x) = \lim\limits_{x\to\infty} \mathrm{e}^{-x/2} \partial_x U(x) = 0. \end{cases} \tag{3.3}$$

Let $\nu_1 = 1$, $\nu_2 = \lambda - \frac{1}{4}$, $U^k(x) = U(x)|_{x\in\Lambda_k}$ and $f^k(x) = f(x)|_{x\in\Lambda_k}$. In particular, $U(x) = U^0(x)$ and $f(x) = f^0(x)$. A weak formulation of (3.3) is to find $U^k(x) \in H^1_{\omega_k}(\Lambda_k)$ such that

$$a^{\nu}_{\omega_k,\Lambda_k}\big(U^k, v\big) = \big(f^k, v\big)_{\omega_k,\Lambda_k}, \quad \forall v \in {}_0H^1_{\omega_k}(\Lambda_k),\ k \in \mathbb{K}, \tag{3.4}$$

with $U^k(a_k) = U(a_k)$. Equivalently, it is to seek $U^k_*(x) = U^k(x) - U^k(a_k) \in {}_0H^1_{\omega_k}(\Lambda_k)$ such that

$$a^{\nu}_{\omega_k,\Lambda_k}\big(U^k_*, v\big) = \big(f^k, v\big)_{\omega_k,\Lambda_k} - a^{\nu}_{\omega_k,\Lambda_k}\big(U^k(a_k), v\big), \quad \forall v \in {}_0H^1_{\omega_k}(\Lambda_k),\ k \in \mathbb{K}. \tag{3.5}$$

If for all $k \in \mathbb{K}$, $f^k(x) \in H^{-1}_{\omega_k}(\Lambda_k)$, then (3.4) has a unique solution such that $U^k(x) \in H^1_{\omega_k}(\Lambda_k)$, $k \in \mathbb{K}$.

Next, let ${}_0\mathcal{P}_{N_k}(\Lambda_k) = {}_0H^1_{\omega_k}(\Lambda_k) \cap \mathcal{P}_{N_k}(\Lambda_k)$, $\nu = (1, \lambda - \frac{1}{4})$ and

$$a^{\nu}_{\omega_k,N_k,\Lambda_k}(u, v) = (\partial_x u, \partial_x v)_{\omega_k,N_k,\Lambda_k} + \left(\lambda - \frac{1}{4}\right)(u, v)_{\omega_k,N_k,\Lambda_k}.$$

The stair Laguerre pseudospectral scheme is to find $u^k_{N_k}(x) \in \mathcal{P}_{N_k}(\Lambda_k)$ such that

$$a^{\nu}_{\omega_k,N_k,\Lambda_k}\big(u^k_{N_k}, \phi\big) = \big(f^k, \phi\big)_{\omega_k,N_k,\Lambda_k}, \quad \forall \phi \in {}_0\mathcal{P}_{N_k}(\Lambda_k),\ k \in \mathbb{K}. \tag{3.6}$$

Equivalently, it is to seek $u^k_{N_k,*}(x) = u^k_{N_k}(x) - u^{k-1}_{N_{k-1}}(a_k) \in {}_0\mathcal{P}_{N_k}(\Lambda_k)$, $k \in \mathbb{K}$ (for $k = 0$, $u^{k-1}_{N_{k-1}}(a_k) = U_0$) such that

$$\begin{aligned} &a^{\nu}_{\omega_k,N_k,\Lambda_k}\big(u^k_{N_k,*}, \phi\big) = \big(f^k, \phi\big)_{\omega_k,N_k,\Lambda_k} - a^{\nu}_{\omega_k,N_k,\Lambda_k}\big(u^{k-1}_{N_{k-1}}(a_k), \phi\big), \\ &\quad \forall \phi \in {}_0\mathcal{P}_{N_k}(\Lambda_k),\ k \in \mathbb{K}. \end{aligned} \tag{3.7}$$

We now turn to estimate the difference between $U^k(x)$ and $u^k_{N_k}(x)$. To do this, let $U^k_{N_k}(x) = P^1_{N_k} U^k(x)$ and $U^k_{N_k,*}(x) = U^k_{N_k}(x) - U^k_{N_k}(a_k)$. Hereafter, we assume that for certain constant c_*, $\sup_{k\in\mathbb{K}}\{a_k - a_{k-1}\} \leqslant c_*$. We have the following convergence result.

Theorem 3.1. Let $\lambda > 1/4$. If for $1 \leqslant k \leqslant K$ and integers $r, s \geqslant 1$, $U^{k-1} \in B^r(\Lambda_{k-1})$, $\partial_x U^k \in A^{r-1}(\Lambda_k)$ and $f^k \in B^s(\Lambda_k)$, then

$$\begin{aligned} \big\|u^k_{N_k} - U^k\big\|_{1,\omega_k,\Lambda_k} \leqslant{} & c\big(\ln N_{k-1}\big)^{1/2} N_{k-1}^{1-r/2} \big|U^{k-1}\big|_{B^r,\Lambda_{k-1}} + cN_k^{1/2-r/2}\big|\partial_x U^k\big|_{A^{r-1},\Lambda_k} \\ & + c(\ln N_k)^{1/2} N_k^{1/2-s/2}\big|f^k\big|_{B^s,\Lambda_k}. \end{aligned} \tag{3.8}$$

In particular,

$$\|u_{N_0} - U\|_{1,\omega,\Lambda} \leqslant cN_0^{1/2-r/2}|\partial_x U|_{A^{r-1},\Lambda} + c(\ln N_0)^{1/2}N_0^{1/2-s/2}|f|_{B^s,\Lambda}. \tag{3.9}$$

Proof. By (2.12), (3.5) and (3.7),

$$\begin{aligned}
&C_1(\lambda)\big\|u^k_{N_k,*} - U^k_{N_k,*}\big\|^2_{1,\omega_k,\Lambda_k}\\
&\quad\leqslant a^\nu_{\omega_k,N_k,\Lambda_k}\big(u^k_{N_k,*} - U^k_{N_k,*}, u^k_{N_k,*} - U^k_{N_k,*}\big)\\
&\quad= \big(f^k, u^k_{N_k,*} - U^k_{N_k,*}\big)_{\omega_k,N_k,\Lambda_k} - a^\nu_{\omega_k,N_k,\Lambda_k}\big(u^{k-1}_{N_{k-1}}(a_k), u^k_{N_k,*} - U^k_{N_k,*}\big)\\
&\qquad - a^\nu_{\omega_k,N_k,\Lambda_k}\big(U^k_{N_k,*}, u^k_{N_k,*} - U^k_{N_k,*}\big) + a^\nu_{\omega_k,\Lambda_k}\big(U^k_*, u^k_{N_k,*} - U^k_{N_k,*}\big)\\
&\qquad + a^\nu_{\omega_k,\Lambda_k}\big(U^k(a_k), u^k_{N_k,*} - U^k_{N_k,*}\big) - \big(f^k, u^k_{N_k,*} - U^k_{N_k,*}\big)_{\omega_k,\Lambda_k}
\end{aligned}$$

where $C_1(\lambda) = \min(1, \lambda - \frac{1}{4})$. Thus, we have that

$$\begin{aligned}
&\big\|u^k_{N_k,*} - U^k_{N_k,*}\big\|_{1,\omega_k,\Lambda_k}\\
&\quad\leqslant \frac{1}{C_1(\lambda)}\left\{\sup_{\phi\in\mathcal{P}^0_{N_k}(\Lambda_k),\ \phi\neq0}\frac{|a^\nu_{\omega_k,\Lambda_k}(U^k_*,\phi) - a^\nu_{\omega_k,N_k,\Lambda_k}(U^k_{N_k,*},\phi)|}{\|\phi\|_{1,\omega_k,\Lambda_k}}\right.\\
&\qquad + \sup_{\phi\in\mathcal{P}^0_{N_k}(\Lambda_k),\ \phi\neq0}\frac{|a^\nu_{\omega_k,\Lambda_k}(U^k(a_k),\phi) - a^\nu_{\omega_k,N_k,\Lambda_k}(u^{k-1}_{N_{k-1}}(a_k),\phi)|}{\|\phi\|_{1,\omega_k,\Lambda_k}}\\
&\qquad \left. + \sup_{\phi\in\mathcal{P}^0_{N_k}(\Lambda_k),\ \phi\neq0}\frac{|(f^k,\phi)_{\omega_k,\Lambda_k} - (f^k,\phi)_{\omega_k,N_k,\Lambda_k}|}{\|\phi\|_{1,\omega_k,\Lambda_k}}\right\}.
\end{aligned}\tag{3.10}$$

We now estimate the three terms at the right-hand side of (3.10). Firstly, by (2.5), (2.8) and lemma 2.3,

$$\begin{aligned}
\big|a^\nu_{\omega_k,\Lambda_k}\big(U^k_*,\phi\big) - a^\nu_{\omega_k,N_k,\Lambda_k}\big(U^k_{N_k,*},\phi\big)\big| &= \left(\lambda - \frac{1}{4}\right)\big|\big(U^k(a_k) - U^k_{N_k}(a_k),\phi\big)_{\omega_k,\Lambda_k}\big|\\
&\leqslant \sqrt{2}\left(\lambda - \frac{1}{4}\right)\big\|U^k - P^1_{N_k}U^k\big\|_{1,\omega_k,\Lambda_k}\|\phi\|_{\omega_k,\Lambda_k}\\
&\leqslant cN_k^{1/2-r/2}\big|\partial_x U^k\big|_{A^{r-1},\Lambda_k}\|\phi\|_{\omega_k,\Lambda_k}.
\end{aligned}\tag{3.11}$$

Similarly, we have from (2.5) and lemma 2.4 that

$$\begin{aligned}
&\big|a^\nu_{\omega_k,\Lambda_k}\big(U^k(a_k),\phi\big) - a^\nu_{\omega_k,N_k,\Lambda_k}\big(u^{k-1}_{N_{k-1}}(a_k),\phi\big)\big|\\
&\quad= \left(\lambda - \frac{1}{4}\right)\big|\big(U^k(a_k) - u^{k-1}_{N_{k-1}}(a_k),\phi\big)_{\omega_k,\Lambda_k}\big|\\
&\quad\leqslant c\big|U^{k-1}(a_k) - u^{k-1}_{N_{k-1}}(a_k)\big|\|\phi\|_{\omega_k,\Lambda_k}\\
&\quad\leqslant c\big|U^{k-1}(a_k) - \mathcal{I}_{N_{k-1}}U^{k-1}(a_k)\big|\|\phi\|_{\omega_k,\Lambda_k}\\
&\quad= c\mathrm{e}^{a_k-a_{k-1}}\big|\big(U^{k-1}(a_k) - \mathcal{I}_{N_{k-1}}U^{k-1}(a_k)\big)\omega_{k-1}(a_k)\big|\|\phi\|_{\omega_k,\Lambda_k}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\Big(\big\|U^{k-1}-\mathcal{I}_{N_{k-1}}U^{k-1}\big\|^{1/2}_{\omega_{k-1},\Lambda_{k-1}}\big|U^{k-1}-\mathcal{I}_{N_{k-1}}U^{k-1}\big|^{1/2}_{1,\omega_{k-1},\Lambda_{k-1}}\\
&\quad+\big\|U^{k-1}-\mathcal{I}_{N_{k-1}}U^{k-1}\big\|_{\omega_{k-1},\Lambda_{k-1}}\Big)\|\phi\|_{\omega_k,\Lambda_k}\\
&\leqslant c(\ln N_{k-1})^{1/2}N_{k-1}^{1-r/2}\big|U^{k-1}\big|_{B^r,\Lambda_{k-1}}\|\phi\|_{\omega_k,\Lambda_k}.
\end{aligned}\tag{3.12}
$$

Furthermore, by (2.12) and lemma 2.4,

$$
\begin{aligned}
\big|\big(f^k,\phi\big)_{\omega_k,\Lambda_k}-\big(f^k,\phi\big)_{\omega_k,N_k,\Lambda_k}\big|&=\big|\big(f^k-\mathcal{I}_{N_k}f^k,\phi\big)_{\omega_k,\Lambda_k}\big|\\
&\leqslant c(\ln N_k)^{1/2}N_k^{1/2-s/2}\big|f^k\big|_{B^s,\Lambda_k}\|\phi\|_{\omega_k,\Lambda_k}.
\end{aligned}
$$

The above with (3.10)–(3.12) gives that

$$
\begin{aligned}
\big\|u^k_{N_k,*}-U^k_{N_k,*}\big\|_{1,\omega_k,\Lambda_k}\leqslant{}& c(\ln N_{k-1})^{1/2}N_{k-1}^{1-r/2}\big|U^{k-1}\big|_{B^r,\Lambda_{k-1}}+cN_k^{1/2-r/2}\big|\partial_xU^k\big|_{A^{r-1},\Lambda_k}\\
&+c(\ln N_k)^{1/2}N_k^{1/2-s/2}\big|f^k\big|_{B^s,\Lambda_k}.
\end{aligned}
$$

Therefore, using (3.11), (3.12), lemmas 2.3 and 2.4, we obtain that for $1\leqslant i\leqslant K$,

$$
\begin{aligned}
\big\|u^k_{N_k}-U^k\big\|_{1,\omega_k,\Lambda_k}&\leqslant\big\|U^k_{N_k}-U^k\big\|_{1,\omega_k,\Lambda_k}+\big\|u^k_{N_k}-U^k_{N_k}\big\|_{1,\omega_k,\Lambda_k}\\
&\leqslant\big\|P^1_{N_k}U^k-U^k\big\|_{1,\omega_k,\Lambda_k}+\big\|u^k_{N_k,*}-U^k_{N_k,*}\big\|_{1,\omega_k,\Lambda_k}\\
&\quad+c\big|U^k_{N_k}(a_k)-U^k(a_k)\big|+c\big|U^k(a_k)-u^{k-1}_{N_{k-1}}(a_k)\big|\\
&\leqslant c(\ln N_{k-1})^{1/2}N_{k-1}^{1-r/2}\big|U^{k-1}\big|_{B^r,\Lambda_{k-1}}+cN_k^{1/2-r/2}\big|\partial_xU^k\big|_{A^{r-1},\Lambda_k}\\
&\quad+c(\ln N_k)^{1/2}N_k^{1/2-s/2}\big|f^k\big|_{B^s,\Lambda_k}.
\end{aligned}
$$

This yields (3.8). Since $u^{k-1}_{N_{k-1}}(a_k)=U^k(a_k)=U_0$, for $k=0$, we find from (3.12) that the second term at the right-hand side of (3.10) vanishes, and so (3.9) follows immediately. □

Remark 3.1. The condition $\lambda>\frac{1}{4}$ is not essential. Indeed, for any $\lambda>0$, we can use a simple variable transformation to reform (3.1) so that λ satisfies the condition in theorem 3.1.

Remark 3.2. For $K=0$, this method is the same as in [10]. But we can use this method with $K>1$ to improve the numerical results, see section 4 of this paper.

In the last theorem, we estimated the numerical error of $u^k_{N_k}(x)$. We now derive an error estimate of the numerical approximation of the original problem (3.1). For this purpose, let

$$
\begin{aligned}
&V^k(x)=\mathrm{e}^{-x/2}U(x)\big|_{x\in\Lambda_k},\qquad F^k(x)=\mathrm{e}^{-x/2}f(x)\big|_{x\in\Lambda_k},\\
&v^k_{N_k}(x)=\mathrm{e}^{-x/2}u^k_{N_k}(x)\big|_{x\in\Lambda_k},\quad k\in\mathbb{K}.
\end{aligned}
$$

In particular, $V(x)=V^0(x)$, $F(x)=F^0(x)$ and $v_{N_0}(x)=v^0_{N_0}(x)$.

By theorem 3.1, we have the following result.

Theorem 3.2. Let $\lambda > \frac{1}{4}$. If for $1 \leqslant k \leqslant K$ and integers $r, s \geqslant 1$, $\mathrm{e}^{x/2}V^{k-1}(x) \in B^r(\Lambda_{k-1})$, $\partial_x(\mathrm{e}^{x/2}V^k(x)) \in A^{r-1}(\Lambda_k)$ and $\mathrm{e}^{x/2}F^k(x) \in B^s(\Lambda_k)$, then

$$\begin{aligned}\left\|v_{N_k}^k - V^k\right\|_{1,\Lambda_k} \leqslant{}& c(\ln N_{k-1})^{1/2}N_{k-1}^{1-r/2}\left|\mathrm{e}^{x/2}V^{k-1}\right|_{B^r,\Lambda_{k-1}} + cN_k^{1/2-r/2}\left|\partial_x\left(\mathrm{e}^{x/2}V^k\right)\right|_{A^{r-1},\Lambda_k}\\ &+ c(\ln N_k)^{1/2}N_k^{1/2-s/2}\left|\mathrm{e}^{x/2}F^k\right|_{B^s,\Lambda_k}.\end{aligned} \tag{3.13}$$

In particular,

$$\|v_{N_0} - V\|_{1,\Lambda} \leqslant cN_0^{1/2-r/2}\left|\partial_x\left(\mathrm{e}^{x/2}V\right)\right|_{A^{r-1},\Lambda} + c(\ln N_0)^{1/2}N_0^{1/2-s/2}\left|\mathrm{e}^{x/2}F\right|_{B^s,\Lambda}. \tag{3.14}$$

Proof. Clearly,

$$\begin{aligned} v_{N_k}^k(x) - V^k(x) &= \mathrm{e}^{-x/2}\left(u_{N_k}^k(x) - U^k(x)\right),\\ \partial_x\left(v_{N_k}^k(x) - V^k(x)\right) &= \mathrm{e}^{-x/2}\partial_x\left(u_{N_k}^k(x) - U^k(x)\right) - \frac{1}{2}\mathrm{e}^{-x/2}\left(u_{N_k}^k(x) - U^k(x)\right).\end{aligned}$$

Hence

$$\left\|v_{N_k}^k - V^k\right\|_{1,\Lambda_k} \leqslant c\left\|u_{N_k}^k - U^k\right\|_{1,\omega_k,\Lambda_k}.$$

The above with theorem 3.1 leads to the desired results. □

In the end of this section, we consider the global solution of the stair Laguerre pseudospectral method and its error estimation. To do this, for any $k \in \mathbb{K}$, let M_k be the number of the nodes x_j^k in the interval $[a_k, a_{k+1}](a_{K+1} = \infty)$ and $M = (M_0, M_1, \ldots, M_K)$. The global numerical solution is given by

$$v_M(x) = v_{N_k}^k(x), \quad x \in [a_k, a_{k+1}],\ k \in \mathbb{K}. \tag{3.15}$$

For simplicity, let

$$\overline{H}^r(\Lambda) = \left\{v \mid \mathrm{e}^{x/2}v(x)\big|_{x\in\Lambda_k} \in B^r(\Lambda_k),\ \forall k \in \mathbb{K}\right\},$$

equipped with the norm

$$|||v|||_{\overline{H}^r(\Lambda)} = \left(\sum_{k\in\mathbb{K}}\left|\mathrm{e}^{x/2}v^k\right|^2_{B^r,\Lambda_k}\right)^{1/2}.$$

As a consequence of theorem 3.2, we have the following results.

Theorem 3.3. Let $V(x)$ be the solution of (3.1), and $v_M(x)$ be the numerical solution given by (3.15). If for all $k \in \mathbb{K}$, $M_k \leqslant N_k$, and for integers $r, s \geqslant 1$, $V(x) \in \overline{H}^r(\Lambda)$, $F(x) \in \overline{H}^s(\Lambda)$, then

$$\|v_M - V\|_{1,\Lambda} \leqslant cG(N, r)\ |||V|||_{\overline{H}^r(\Lambda)} + c\widetilde{G}(N, s)\ |||F|||_{\overline{H}^s(\Lambda)} \tag{3.16}$$

where

$$G(N,r)=\Bigg(\sum_{k\in\mathbb{K}}(\ln N_k)N_k^{2-r}\Bigg)^{1/2},\qquad \widetilde{G}(N,s)=\Bigg(\sum_{k\in\mathbb{K}}(\ln N_k)N_k^{1-s}\Bigg)^{1/2}.$$

Proof. By (3.15) and theorem 3.2,

$$\begin{aligned}\|v_M-V\|_{1,\Lambda}&\leqslant\sum_{k\in\mathbb{K}}\big\|v_{N_k}^k-V^k\big\|_{1,[a_k,a_{k+1}]}\leqslant\sum_{k\in\mathbb{K}}\big\|v_{N_k}^k-V^k\big\|_{1,\Lambda_k}\\&\leqslant cG(N,r)\,|||V|||_{\overline{H}^r(\Lambda)}+c\widetilde{G}(N,s)\,|||F|||_{\overline{H}^s(\Lambda)}\,.\end{aligned}$$

This completes the proof. □

Finally, we give a pointwise error estimation of the global stair Laguerre pseudospectral method. To do this, let

$$\widetilde{H}^r(\Lambda)=\big\{v\mid \mathrm{e}^{x/2}v(x)\big|_{x\in\Lambda_k}\in B^r(\Lambda_k),\ \forall k\in\mathbb{K},\ \text{and}\ |||v|||_{\widetilde{H}^r(\Lambda)}<\infty\big\},$$

equipped with the norm

$$|||v|||_{\widetilde{H}^r(\Lambda)}=\max_{k\in\mathbb{K}}\big\{\big|\mathrm{e}^{x/2}v^k\big|_{B^r,\Lambda_k}\big\}.$$

Theorem 3.4. Let $V(x)$ be the solution of (3.1), and $v_M(x)$ be the numerical solution given by (3.15). If for all $k\in\mathbb{K}$, $M_k\leqslant N_k$, and for integers $r,s\geqslant 1$, $V(x)\in\widetilde{H}^r(\Lambda)$, $F(x)\in\widetilde{H}^s(\Lambda)$, then

$$\sup_{x\in\bar{\Lambda}}\big|(v_M-V)(x)\big|\leqslant cW(N,r)\,|||V|||_{\widetilde{H}^r(\Lambda)}+c\widetilde{W}(N,s)\,|||F|||_{\widetilde{H}^s(\Lambda)}\tag{3.17}$$

where

$$\begin{aligned}W(N,r)&=\max_{k\in\mathbb{K}}\big\{\mathrm{e}^{-a_k/2}(\ln N_k)^{1/2}N_k^{1-r/2}\big\},\\\widetilde{W}(N,s)&=\max_{k\in\mathbb{K}}\big\{\mathrm{e}^{-a_k/2}(\ln N_k)^{1/2}N_k^{1/2-s/2}\big\}.\end{aligned}$$

Proof. By lemma 2.1,

$$\begin{aligned}\sup_{x\in\bar{\Lambda}_k}\big|\big(v_{N_k}^k-V^k\big)(x)\big|&\leqslant\mathrm{e}^{-a_k/2}\sup_{x\in\bar{\Lambda}_k}\big|\big(u_{N_k}^k(x)-U^k(x)\big)\omega_k^{1/2}(x)\big|\\&\leqslant\sqrt{2}\mathrm{e}^{-a_k/2}\big\|u_{N_k}^k-U^k\big\|_{1,\omega_k,\Lambda_k},\quad k\in\mathbb{K}.\end{aligned}$$

Thus, by this fact and theorem 3.2,

$$\begin{aligned}\sup_{x\in\bar{\Lambda}}\big|(v_M-V)(x)\big|&=\max_{k\in\mathbb{K}}\sup_{x\in[a_k,a_{k+1}]}\big|\big(v_{N_k}^k-V\big)(x)\big|\leqslant\max_{k\in\mathbb{K}}\sup_{x\in\bar{\Lambda}_k}\big|\big(v_{N_k}^k-V\big)(x)\big|\\&\leqslant c\max_{k\in\mathbb{K}}\big\{\mathrm{e}^{-a_k/2}\big\|u_{N_k}^k-U^k\big\|_{1,\omega_k,\Lambda_k}\big\}\leqslant cW(N,r)\,|||V|||_{\widetilde{H}^r(\Lambda)}\\&\quad+c\widetilde{W}(N,s)\,|||F|||_{\widetilde{H}^s(\Lambda)}\,.\end{aligned}$$

This ends the proof. □

4. Numerical results

This section is for some numerical results. We first use scheme (3.7) to solve the reformed problem (3.3) numerically. As in [10], we introduce the base functions

$$\phi_l^k(x) = \mathcal{L}_l^k(x) - \mathcal{L}_{l+1}^k(x), \quad k \in \mathbb{K}. \tag{4.1}$$

Since $\mathcal{L}_l^k(a_k) = 1$ for all l, we have $\phi_l^k(a_k) = 0$ and so

$$_0\mathcal{P}_{N_k}(\Lambda_k) = \operatorname{span}\{\phi_0^k, \phi_1^k, \ldots, \phi_{N_k-1}^k\}, \quad k \in \mathbb{K}.$$

Furthermore, by (2.2),

$$\partial_x \phi_l^k(x) = \partial_x \mathcal{L}_l^k(x) - \partial_x \mathcal{L}_{l+1}^k(x) = \mathcal{L}_l^k(x).$$

Thus, by (2.3) and (2.12),

$$\left(\partial_x \phi_l^k, \partial_x \phi_m^k\right)_{\omega_k, N_k, \Lambda_k} = \left(\partial_x \phi_l^k, \partial_x \phi_m^k\right)_{\omega_k, \Lambda_k} = \left(\mathcal{L}_l^k, \mathcal{L}_m^k\right)_{\omega_k, \Lambda_k} = \delta_{l,m}.$$

Now, let

$$u_{N_k,*}^k(x) = \sum_{l=0}^{N_k-1} \hat{h}_l^k \phi_l^k(x), \quad k \in \mathbb{K},$$

and

$$\begin{aligned}
&\widehat{\mathbf{H}}^k = \left(\hat{h}_0^k, \hat{h}_1^k, \ldots, \hat{h}_{N_k-1}^k\right)^{\mathrm{T}},\\
&\mathbf{R}^k = \left(r_0^k, r_1^k, \ldots, r_{N_k-1}^k\right)^{\mathrm{T}}, \quad r_l^k = \left(f^k, \phi_l^k\right)_{\omega_k, N_k, \Lambda_k} - a_{\omega_k, N_k, \Lambda_k}^{\nu}\left(u_{N_{k-1}}^{k-1}(a_k), \phi_l^k\right),\\
&\mathbf{B} = (b_{lj})_{l,j=0,\ldots,N_k-1}, \qquad b_{lj} = \left(\phi_j^k, \phi_l^k\right)_{\omega_k, \Lambda_k}.
\end{aligned}$$

In view of (2.3) and (4.1), we have that

$$b_{lj} = \begin{cases} 2, & l = j, \\ -1, & l = j-1, j+1, \\ 0, & \text{otherwise}. \end{cases}$$

Accordingly, the scheme (3.7) is equivalent to

$$\left(\mathbf{I} + \left(\lambda - \frac{1}{4}\right)\mathbf{B}\right)\widehat{\mathbf{H}}^k = \mathbf{R}^k, \quad k \in \mathbb{K}. \tag{4.2}$$

Therefore, at each stair step $k \in \mathbb{K}$, we only need to resolve the symmetric and tridiagonal system (4.2), and then $u_{N_k}^k(x) = u_{N_k,*}^k + u_{N_{k-1}}^{k-1}(a_k)$ and $v_{N_k}^k(x) = \mathrm{e}^{-x/2} u_{N_k}^k(x)$. This feature simplifies computation and saves a lot of work.

We now present some numerical results. We take the following exact solution of (3.1),

$$V(x) = \mathrm{e}^{-(x-x_0)^2/h}, \quad x \in \Lambda, \ h > 0. \tag{4.3}$$

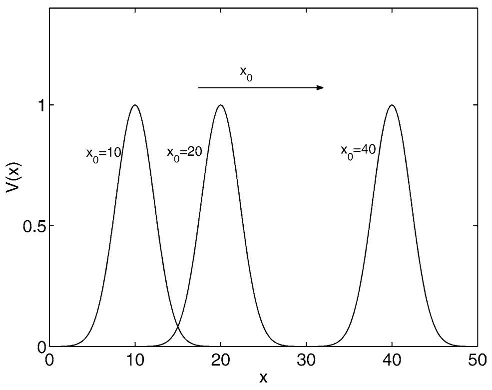

Figure 2. $V(x)$ with $h = 10$ and various x_0.

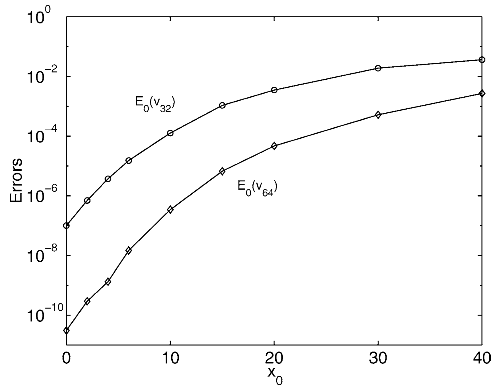

Figure 3. The errors for $K = 0$ and various x_0.

As shown in figure 2, the interested information of $V(x)$ is mainly contained in the interval $\Delta(x_0, h) = [x_0 - 3\sqrt{h/2},\ x_0 + 3\sqrt{h/2}]$, where $V(x)$ changes rapidly. For description of the errors, let

$$E_K(v_M) = \max_{k\in\mathbb{K}} \max_{0\leqslant j\leqslant M_k} \left|v_M\left(x_j^k\right) - V\left(x_j^k\right)\right|.$$

We first take $h = 10$ and $x_0 \in [0, 40]$ in (4.3). The errors of the single-stair method ($K = 0$ in (3.7)) with $M_0 = N_0 = 32$, and $M_0 = N_0 = 64$ are illustrated in figure 3. It can be seen that the errors increase very fast as the center x_0 of the interested interval $\Delta(x_0, h)$ moves away from the left endpoint. This is mainly due to the locations of the Laguerre–Gauss–Radau quadrature nodes. As shown in figure 1, the nodes are relatively denser near the left endpoint, and the distances between adjacent nodes grow very quickly as x and the number of nodes increase.

We next use the two-stair method to solve (3.1) numerically. We plot in figure 4 the errors of the two-stair method with (i) $N_0 = N_1 = 32, M_0 = 16, M_1 = 32, a_0 = 0, a_1 = x_{16}^0$, and (ii) $N_0 = N_1 = 64, M_0 = 22, M_1 = 64, a_0 = 0, a_1 = x_{22}^0$. It shows that the two-stair approximation provides better numerical results than the single-stair approximation.

We now take $h = 10$ and $x_0 = 95$ in (4.3). The errors of the single-stair method with $M_0 = N_0 = 32, 40, 45, 50, 60, 70, 80, 90$, and the errors of the two-stair method with $N_0 = 32, M_0 = 30, a_0 = 0, a_1 = x_{29}^0$ and $M_1 = N_1 = 32, 40, 45, 50, 60, 70, 80, 90$, are illustrated in figure 5. It shows that the single-stair scheme can not provide good numerical solutions when the interested interval $\Delta(x_0, h)$ is big and far from the left endpoint. However, the multiple-stair Laguerre scheme provides very accurate numerical results. For example, the errors of single-stair approximation $E_0(v_{(60)}) = 8.16\times 10^{-2}$, $E_0(v_{(90)}) = 2.42\times 10^{-2}$, but the errors of two-stair approximation ($N_0 = 32$, $M_0 = 30, a_0 = 0, a_1 = x_{29}^0$) $E_1(v_{(30,32)}) = 4.36 \times 10^{-3}$, $E_1(v_{(30,60)}) = 9.80 \times 10^{-6}$. Thus multiple stair Laguerre pseudospectral method not only saves the work, but also raises the accuracy. Figure 5 also indicates the rapid convergence of this new method.

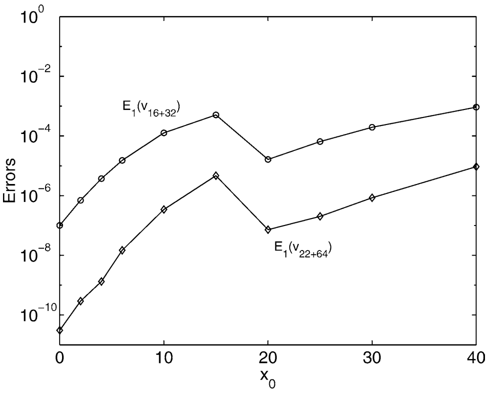

Figure 4. The errors for $K = 1$ and various x_0.

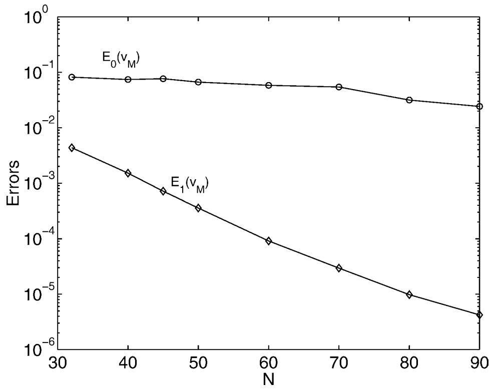

Figure 5. The errors for $K = 0$ and $K = 1$.

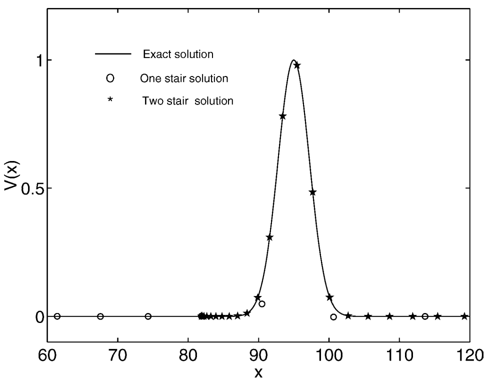

Figure 6. $V(x)(h = 10, x_0 = 95)$ vs. $v_{32}^1(x)$.

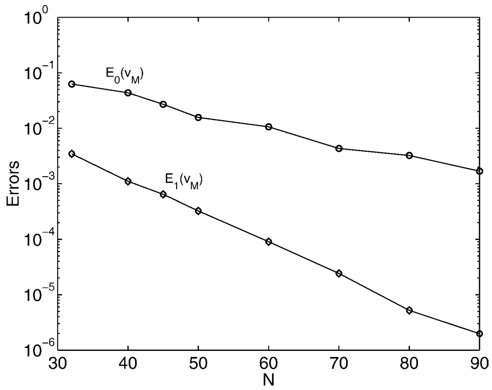

Figure 7. The errors for $K = 0$ and $K = 1$.

The stair Laguerre pseudospectral method can also be used to improve the resolution of the standard Laguerre approximation locally. For instance, we plot in figure 6 the exact solution $V(x)$ ($h = 10, x_0 = 95, x \in [60, 120]$) vs. the numerical solution by the single-stair method with $M_0 = N_0 = 32$, and the numerical solution by the two-stair method with $N_0 = 32$, $M_0 = 30, a_0 = 0, a_1 = x_{30}^0$, and $M_1 = N_1 = 32$. Clearly, the single-stair method cannot identify the interested structure of the exact solution in the subinterval [90, 100]. But the two-stair method recovers the interested structure of $V(x)$ in this subinterval very well. It is noted that we usually take a_k as some interpolation points near the subintervals where the numerical solutions by the single-stair method change very rapidly. In this example, there exists one peak. So we only need to use the two-stair method with $a_0 = 0$ and $a_1 = x_{30}^0$. Clearly the work is only double of the work in the single-stair method.

This technique is also available for the solutions with several peaks. For instance, we take the exact solution of (3.1) with two peaks

$$V(x) = \mathrm{e}^{-(1-x_1)^2/h_1} + \mathrm{e}^{-(1-x_2)^2/h_2}, \quad x \in \Lambda.$$

Let $x_1 = 10$, $x_2 = 50$, $h_1 = 20$ and $h_2 = 10$. The errors of the single-stair method with $M_0 = N_0 = 32, 40, 45, 50, 60, 70, 80, 90$, and the errors of the two-stair method with $N_0 = 32$, $M_0 = 20$, $a_0 = 0$, $a_1 = x^0_{19}$ and $M_1 = N_1 = 32, 40, 45, 50, 60, 70, 80, 90$, are illustrated in figure 7. It indicates that the multiple-stair scheme provides more accurate numerical solution than the standard Laguerre pseudospecctral method.

5. Concluding remarks

Since the distances between the Laguerre interpolation nodes increase very fast as the mode N increases, the numerical solutions cannot simulate the exact solutions properly for large x. In this paper, we first use small mode N_0 to obtain accurate numerical solutions at the nodes which are not far from the original point $x = 0$, and located almost uniformly. Thus we obtain better numerical results at those nodes. Then we use the shift Laguerre pseudospectral approximation to extend the numerical results for large x, step by step. Because at each step, the mode N_k is also small, so we also obtain accurate numerical results for large x. The numerical results coincide the analysis. On the other hand, using small N_k at each step, simplifies the computation and saves work.

For simplicity, we only take a simple model problem as an example to show how the new algorithm works well. But it is not difficult to generalize this method to nonlinear problems, evolutionary problems and multiple-dimensional problems in unbounded domains.

Appendix. The proof of lemma 2.2

Proof. Let $\mathcal{L}^{(\alpha)}_l(x)$ be the generalized Laguerre polynomials of degree l. We know from [16] that for $\alpha > -1$,

$$\partial_x \mathcal{L}^{(\alpha)}_l(x) = -\partial_x \mathcal{L}^{(\alpha+1)}_{l-1}(x), \quad l = 1, 2, \ldots, \tag{A.1}$$

and

$$\int_\Lambda \mathcal{L}^{(\alpha)}_l(x)\mathcal{L}^{(\alpha)}_m(x) x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x = \gamma^{(\alpha)}_l \delta_{l,m}, \tag{A.2}$$

where

$$\gamma^{(\alpha)}_l = \frac{\Gamma(l+\alpha+1)}{\Gamma(l+1)}. \tag{A.3}$$

Next, let a_k, ω_k and Λ_k be the same as in section 2. Denote $\mathcal{L}^{(\alpha)}_{l,k}(x) = \mathcal{L}^{(\alpha)}_l(x - a_k)$, $x \in \Lambda_k$. In particular, $\mathcal{L}_{l,k}(x) = \mathcal{L}^{(0)}_{l,k}(x)$. Clearly, for any $v \in L^2_{\omega_k}(\Lambda_k)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l \mathcal{L}_{l,k}(x), \quad \hat{v}_l = (v, \mathcal{L}_{l,k})_{\omega_k,\Lambda_k}.$$

Thus, by (A.1),

$$\partial_x^j v(x) = \sum_{l=j}^{\infty} \hat{v}_l \partial_x^j \mathcal{L}_{l,k}(x) = \sum_{l=j}^{\infty} (-1)^j \hat{v}_l \mathcal{L}_{l-j,k}^{(j)}(x). \tag{A.4}$$

Moreover, by (A.2),

$$\left\| (x-a_k)^{j/2} \mathcal{L}_{l-j,k}^{(j)} \right\|_{\omega_k,\Lambda_k}^2 = \gamma_{l-j}^{(j)}.$$

Using (A.2) again, we obtain that

$$\begin{aligned} \| P_{N_k} v - v \|_{\omega_k,\Lambda_k}^2 &= \sum_{l=N_k+1}^{\infty} \hat{v}_l^2 \leqslant \max_{l \geqslant N_k+1} \left(\gamma_{l-r}^{(r)}\right)^{-1} \sum_{l=r}^{\infty} \hat{v}_l^2 \gamma_{l-r}^{(r)} \\ &= \max_{l \geqslant N_k+1} \left(\gamma_{l-r}^{(r)}\right)^{-1} \left\| (x-a_k)^{r/2} \partial_x^r v \right\|_{\omega_k,\Lambda_k}^2. \end{aligned} \tag{A.5}$$

Therefore, by using (A.3) and the Stirling formula,

$$\Gamma(s+1) = \sqrt{2\pi s}\, s^s \mathrm{e}^{-s}\left(1 + \mathrm{O}\left(s^{-1/5}\right)\right),$$

we get that

$$\max_{l \geqslant N_k+1} \left(\gamma_{l-r}^{(r)}\right)^{-1} = \frac{\Gamma(N_k - r + 1)}{\Gamma(N_k + 2)} \sim N_k^r. \tag{A.6}$$

Finally, a combination of (A.5) and (A.6) leads to the desired result. □

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