Appl. Comput. Harmon. Anal. 29 (2010) 303–329

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

A new generalization of the PSWFs with applications to spectral approximations on quasi-uniform grids ☆

Li-Lian Wang *, Jing Zhang

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore

ARTICLE INFO

Article history:
Received 15 November 2008
Revised 18 August 2009
Accepted 10 December 2009
Available online 16 December 2009
Communicated by Gregory Beylkin

Keywords:
Slepian functions
Generalized prolate spheroidal wave functions
Approximation by GPSWFs
GPSWF spectral collocation and Galerkin methods

ABSTRACT

We define a new family of generalized prolate spheroidal wave functions (GPSWFs), which extends the prolate spheroidal wave functions of order zero (PSWFs or Slepian functions; Slepian and Pollak, 1961 [45]) to real order $\alpha > -1$, and also generalizes the Gegenbauer polynomials to an orthogonal system with an intrinsic tuning parameter $c > 0$. We show that the GPSWFs, defined as the eigenfunctions of a Sturm–Liouville problem, are also the eigenfunctions of an integral operator. We present a number of analytic and asymptotic formulae for the GPSWFs and the associated eigenvalues, and introduce efficient algorithms for their evaluations. Moreover, we derive a set of optimal results on the GPSWF approximations featured with explicit dependence on the parameter c. As an important application, we implement and analyze the GPSWF spectral methods for elliptic-type equations. We illustrate that the presence of c provides flexibility to design high-order approximations on quasi-uniform grids, and endows the GPSWFs with some favorable advantages over their polynomial counterparts.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

About half a century ago, D. Slepian and H.O. Pollak [45] discovered that the prolate spheroidal wave functions of order zero (PSWFs) are bandlimited and maximally concentrated within a given time interval. The heart of this discovery is that the PSWFs, denoted by $\{\psi_n(x;c)\}_{n=0}^{\infty}$, which are the eigenfunctions of the Sturm–Liouville problem

$$\partial_x\big((1-x^2)\partial_x\psi_n(x;c)\big)+\big(\chi_n(c)-c^2x^2\big)\psi_n(x;c)=0,\quad c>0,\ x\in I:=(-1,1), \tag{1.1}$$

also form the eigen-system of the integral equation

$$\lambda_n(c)\psi_n(x;c)=\int_{-1}^{1}e^{\mathrm{i}cxt}\psi_n(t;c)\,dt,\quad c>0,\ x\in I. \tag{1.2}$$

Attributed to D. Slepian, the PSWFs are also termed as the Slepian functions or basis (cf. [25,36] and the interested readers may also refer to [22] and the original references therein for other independent works on the discovery of prolate spheroidal functions). A series of papers by Slepian et al. [45,33,34,44,46] and recent work by Xiao and Rokhlin et al. [55,42] have shown that the Slepian functions are an optimal tool for approximating bandlimited functions. Over the last few years,

☆ This work is partially supported by Singapore AcRF Tier 1 Grant RG58/08, Singapore MOE Grant T207B2202, and Singapore NRF 2007IDM-IDM002-010.
* Corresponding author.
E-mail address: lilian@ntu.edu.sg (L.-L. Wang).

1063-5203/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.acha.2009.12.002

there has been a growing interest in various aspects of the Slepian functions including analytic and asymptotic studies, quadrature and interpolation [57,55,6,56,10,38], and numerical evaluations [12,52,20,30]. Moreover, there occurs a surge of recent activities in developing methods based on this basis [51,50,17,48,11,14,31,7,32]. Most notably, spectral approximations using this bandlimited basis enjoy some remarkable advantages over the Legendre polynomial-based methods [11,14]: (i) enable fewer points per wavelength to resolve waves; (ii) use quasi-uniformly distributed collocation points allowing for larger time step in explicit time-marching schemes; and (iii) achieve a better resolution near the center of the computational domain. The approximation of bandlimited functions by the Slepian functions was analyzed in [42,55], and estimates for the truncation errors for approximating general functions in Sobolev spaces were carried out in [11,14,53]. The results in [53] are optimal and improved the existing estimates, and also confirmed the conjecture in [9] that a super-geometric convergence can be achieved when the Slepian functions are adopted to approximate bandlimited functions.

The extensions of Slepian's time-frequency concentration problems in a variety of geometries and/or the generalizations of the Slepian functions in different senses have subsequently attracted many attentions. D. Slepian [44] extended the earlier works [45,33] to multidimensional settings, and derived a family of generalized PSWFs from the finite Fourier transform on a unit disk. G. Beylkin et al. [5] explored some interesting issues of bandlimited functions in a disk. F. Simons et al. [43] gave an up-to-date review and study of time-frequency and time-scale concentration problems on a sphere. Another important aspect related to Slepian's seminal papers is the investigation of differential operators that commute with appropriate integral operators in the time-and-band limiting context, and we particularly refer to the insightful analysis by F. Grünbaum et al. [24,23,22,25]. Recently, A. Zayed [58] showed by using the theory of reproducing-kernel Hilbert spaces that there are other systems possessing a double orthogonality as the Slepian functions, and in turn the Slepian functions turn out to be a lucky special case under this framework. The work [37] extended (1.2) to a finite fractional Fourier transform with applications to simultaneously time and band concentrating signals in optical systems.

We propose a different generalization of the Slepian functions with a more specific aim to introduce a new family of orthogonal bases for spectral approximations, which overcome some drawbacks of the usual polynomial-based spectral methods. Our generalization is based on two observations. The first observation is related to the Mathieu functions, which are the eigenfunctions of the angular Mathieu equation:

$$v''(\theta) + (\rho - 2q\cos 2\theta)v(\theta) = 0, \quad \theta \in (0, 2\pi),\ q > 0, \tag{1.3}$$

arisen from separating variables for solving the Helmholtz equation in an elliptic domain (cf. [35]). The angular Mathieu functions are periodic and orthogonal in $L^2(0, 2\pi)$, and serve as a natural basis for spectral approximations of scattering problems with ellipse-shaped obstacles [41]. Setting $x = -\cos x$ and $u(x) = v(\theta)$, we transform (1.3) into

$$\big(1 - x^2\big)^{1/2} \frac{d}{dx}\left(\big(1 - x^2\big)^{1/2} \frac{d}{dx} u(x)\right) + \big(\chi - c^2x^2\big)u(x) = 0, \quad x \in (-1, 1),\ c > 0, \tag{1.4}$$

where $\chi = \rho + 2q$ and $c^2 = 4q$. We define the eigenfunctions of the Sturm–Liouville problem (1.4) as the GPSWFs of order $-\frac{1}{2}$, denoted by $\{\psi_n^{(-1/2)}(x; c)\}_{n=0}^{\infty}$, and they form a complete orthogonal system in $L^2_\omega(-1, 1)$ with $\omega = (1 - x^2)^{-1/2}$. This new family can be regarded as a generalization of the Chebyshev polynomials with a tuning parameter c. As with the Slepian functions, they are also "bandlimited" in a weighted sense:

$$\lambda_n^{(-1/2)}(c)\psi_n^{(-1/2)}(x; c) = \int_{-1}^{1} e^{\mathrm{i}cxt}\psi_n^{(-1/2)}(t; c)\frac{1}{\sqrt{1 - t^2}}\,dt, \quad c > 0. \tag{1.5}$$

The second observation is relevant to the angular spheroidal functions (cf. [18]) satisfying

$$\frac{d}{dx}\left(\big(1 - x^2\big)\frac{dv}{dx}\right) + \left(\rho - c^2x^2 - \frac{m^2}{1 - x^2}\right)v = 0, \quad x \in (-1, 1),\ c > 0,\ m = 0, 1, \ldots. \tag{1.6}$$

The substitution: $v(x) = (1 - x^2)^{m/2}u(x)$ leads to the equation in u:

$$\big(1 - x^2\big)\frac{d^2u}{dx^2} - 2(m + 1)x\frac{du}{dx} + \big(\chi - c^2x^2\big)u = 0, \quad c > 0,\ \chi = \rho - m(m + 1),$$

which can be written as the self-adjoint form:

$$\big(1 - x^2\big)^{-m} \frac{d}{dx}\left(\big(1 - x^2\big)^{m+1} \frac{d}{dx} u(x)\right) + \big(\chi - c^2x^2\big)u(x) = 0, \quad c > 0,\ x \in (-1, 1). \tag{1.7}$$

Accordingly, we define the eigenfunctions of this problem as the GPSWFs of integer order m, and show that they satisfy an integral equation similar to (1.5).

Based on the foregoing observations, it is desirable to define a family of GPSWFs of real order $\alpha > -1$, and to explore their properties and applications to numerical approximations of PDEs. Defined as the eigenfunctions of a second-order Sturm–Liouville operator, the GPSWFs form a complete orthogonal system in $L^2_{\omega_\alpha}(-1, 1)$ with $\omega_\alpha(x) = (1 - x^2)^\alpha$, and they are also the eigenfunctions of some integral operators (cf. (3.10) and (3.11) below), which commute with the Sturm–Liouville operator. The GPSWFs are "bandlimited" in a weighted sense, and can be viewed as a generalization of the Gegenbauer polynomials with a tuning parameter c. Such an extension might appear to be natural, but to the best of our knowledge, it has never (at least systematically) been explored in the literature. The relevance lies in several aspects. Firstly, with a suitable choice of c, the GPSWFs oscillate nearly uniformly (cf. Fig. 3.1). This suggests that the grid points associated with the GPSWF basis will be quasi-uniform (cf. Fig. 4.1), and non-uniform resolution of the Gegenbauer expansions will be replaced by a more uniform approximation. Indeed, many problems are only well-posed in certain weighted formulation, so it is necessary to use GPSWF basis with $\alpha \neq 0$. It is anticipated that spectral approximations using GPSWFs have some attractive advantages as listed in (i)–(iii) for spectral methods using the Slepian functions. Moreover, the use of GPSWFs will enhance and enrich the applicability of the Gegenbauer polynomials, which have been widely used in mathematical analysis and numerical approximations (see, e.g., [21,26,19]). Secondly, we realize that it is necessary to put the Slepian functions in a general setting, which might lead to a more precise and concise analysis, and improved results. This is reminiscent to the study of Chebyshev and Legendre polynomials under the general framework for Gegenbauer or Jacobi polynomials (cf. [4,13]). On the other hand, the GPSWFs are closely related to some other special functions such as the Mathieu functions and angular spheroidal functions, so the investigation of GPSWFs will provide new results for these relevant functions.

The rest of the paper is organized as follows. In Section 2, we collect some relevant properties of the Gegenbauer polynomials and the Bessel functions to be used throughout the paper. In Section 3, we define the GPSWFs and study their analytic and asymptotic properties. We analyze the approximation properties of the GPSWFs in weighted Sobolev spaces in Section 4, and implement the GPSWF spectral methods in Section 5. The final section is for some concluding remarks.

2. Mathematical preliminaries

In this section, we introduce some notation and review the relevant properties of the Gegenbauer polynomials (cf. [47]) and the Bessel functions (cf. [54]).

2.1. Notation

- Let $\omega_\alpha(x) = (1-x^2)^\alpha$ $(\alpha > -1)$ be the Gegenbauer weight function defined in $I := (-1, 1)$, and let $L^2_{\omega_\alpha}(I)$ be the Hilbert space with the inner product and norm

$$(u, v)_{\omega_\alpha} = \int_I u(x)v(x)\omega_\alpha(x)\,dx, \qquad \|u\|_{\omega_\alpha} = \sqrt{(u, u)_{\omega_\alpha}}.$$

For any integer $r \geqslant 0$, we define the weighted Sobolev space:

$$H^r_{\omega_\alpha}(I) = \left\{u \in L^2_{\omega_\alpha}(I)\colon u^{(k)} \in L^2_{\omega_\alpha}(I),\ 0 \leqslant k \leqslant r\right\},$$

equipped with the norm and semi-norm:

$$\|u\|_{r,\omega_\alpha} = \left(\sum_{k=0}^{r} \left\|u^{(k)}\right\|^2_{\omega_\alpha}\right)^{\frac{1}{2}}, \qquad |u|_{r,\omega_\alpha} = \left\|u^{(r)}\right\|_{\omega_\alpha}.$$

For any real $r > 0$, the space $H^r_{\omega_\alpha}(I)$ and its norm $\|\cdot\|_{r,\omega_\alpha}$ are defined by space interpolation as in [2].

- We always assume that $\alpha > -1$ and the tuning parameter $c > 0$, and sometimes suppress the dependence of the parameters in the notations, e.g., (a_n, b_n) in (2.4), and $\mathcal{D}_x = \mathcal{D}_x(\alpha, c)$ in (3.1) below.
- We use the notation $A \simeq B$ to mean that for $B \neq 0$, the ratio $\frac{A}{B} \to 1$ in the sense of some limiting process.
- We use $\partial_x^k u(x)$ to denote the ordinary derivative $\frac{d^k}{dx^k}u(x) = u^{(k)}(x)$ for $k \geqslant 1$.

2.2. Gegenbauer polynomials

The Gegenbauer polynomials, denoted by $G_n^{(\alpha)}(x)$, are the eigenfunctions of the Sturm–Liouville problem

$$\mathcal{L}_x^{(\alpha)}\left[G_n^{(\alpha)}\right](x) := -\frac{1}{\omega_\alpha(x)}\partial_x\left(\omega_{\alpha+1}(x)\partial_x G_n^{(\alpha)}(x)\right) = \gamma_n^{(\alpha)} G_n^{(\alpha)}(x), \quad x \in I, \tag{2.1}$$

with the corresponding eigenvalues $\gamma_n^{(\alpha)} = n(n+2\alpha+1)$. They are mutually orthogonal with respect to ω_α, and normalized so that

$$\int_{-1}^{1} G_m^{(\alpha)}(x) G_n^{(\alpha)}(x)\omega_\alpha(x)\,dx = \delta_{mn}, \tag{2.2}$$

where δ_{mn} is the Kronecker symbol.

The normalized Gegenbauer polynomials satisfy the three-term recurrence relation:

$$G_{n+1}^{(\alpha)}(x) = a_n x G_n^{(\alpha)}(x) - b_n G_{n-1}^{(\alpha)}(x), \quad n \geqslant 1,$$
$$G_0^{(\alpha)}(x) = \sqrt{\frac{\Gamma(\alpha+3/2)}{\sqrt{\pi}\,\Gamma(\alpha+1)}}, \qquad G_1^{(\alpha)}(x) = \sqrt{2\alpha+3}\, x G_0^{(\alpha)}(x), \tag{2.3}$$

where $\Gamma(\cdot)$ is the Gamma function, and

$$a_n = \sqrt{\frac{(2n+2\alpha+1)(2n+2\alpha+3)}{(n+1)(n+2\alpha+1)}}, \qquad b_n = \sqrt{\frac{n(n+2\alpha)(2n+2\alpha+3)}{(n+1)(n+2\alpha+1)(2n+2\alpha-1)}}. \tag{2.4}$$

Moreover, we have the formula:

$$\partial_x G_{n+1}^{(\alpha)}(x) = \tilde{a}_n G_n^{(\alpha)}(x) + \tilde{b}_n \partial_x G_{n-1}^{(\alpha)}(x), \quad n \geqslant 1, \tag{2.5}$$

where

$$\tilde{a}_n = \sqrt{\frac{(n+1)(2n+2\alpha+1)(2n+2\alpha+3)}{n+2\alpha+1}}, \qquad \tilde{b}_n = \sqrt{\frac{n(n+1)(2n+2\alpha+3)}{(n+2\alpha)(n+2\alpha+1)(2n+2\alpha-1)}}. \tag{2.6}$$

The Gegenbauer polynomials are even for even n and odd for odd n; namely,

$$G_n^{(\alpha)}(-x) = (-1)^n G_n^{(\alpha)}(x), \quad x \in I, \tag{2.7}$$

and the leading coefficient of $G_n^{(\alpha)}(x)$ is

$$k_n^{(\alpha)} = \frac{2^{n+\alpha+1}\Gamma(n+\alpha+3/2)}{\sqrt{2\pi(2n+2\alpha+1)n!\,\Gamma(n+2\alpha+1)}}. \tag{2.8}$$

One verifies readily that

$$\int_{-1}^{1} x^n G_n^{(\alpha)}(x)\omega_\alpha(x)\,dx = \frac{1}{k_n^{(\alpha)}}, \qquad \int_{-1}^{1} x\partial_x G_n^{(\alpha)}(x) G_n^{(\alpha)}(x)\omega_\alpha(x)\,dx = n. \tag{2.9}$$

2.3. Bessel functions

Let $J_\nu(z)$ $(z>0)$ be the Bessel functions of the first kind of real order ν. We have the following recurrence relation:

$$J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z) = J_\nu'(z) = \frac{\nu}{z}J_\nu(z) - J_{\nu+1}(z), \tag{2.10}$$

which implies

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z). \tag{2.11}$$

The Bessel functions satisfy the Poisson integral formula:

$$J_\nu(z) = \frac{z^\nu}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1} e^{\mathrm{i}zt}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,dt, \quad \nu > -\frac{1}{2}. \tag{2.12}$$

Moreover, for fixed $z>0$ and large ν, there exists the asymptotic formula:

$$J_\nu(z) \simeq \frac{1}{\sqrt{2\pi}}\exp\left(\nu + \nu\ln\left(\frac{z}{2}\right) - \left(\nu+\frac{1}{2}\right)\ln\nu\right). \tag{2.13}$$

The following formula will be useful for the analysis in the forthcoming section.

Lemma 2.1. *Let $G_n^{(\alpha)}(\frac{1}{\mathrm{i}}\frac{d}{dz})$ be the differential operator generated by the Gegenbauer polynomial $G_n^{(\alpha)}(\cdot)$. Then for all $\alpha > -1$ and $n \geqslant 0$,*

$$G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] = \mathrm{i}^n h_n^{(\alpha)}\frac{J_{n+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}, \quad n \geqslant 0,\ z>0, \tag{2.14}$$

where

$$h_n^{(\alpha)} = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\sqrt{\frac{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+1)}}. \tag{2.15}$$

We postpone the proof to Appendix A. To the best of our knowledge, this formula is not even available in the popular books on Bessel and special functions [1,54], though the following formula of similar type can be found

$$\left(\frac{1}{z}\frac{d}{dz}\right)^n\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] = \frac{J_{n+\alpha+\frac{1}{2}}(z)}{z^{n+\alpha+\frac{1}{2}}}, \quad n \geqslant 0,\ z > 0. \tag{2.16}$$

3. Generalized prolate spheroidal wave functions

In this section, we define the GPSWFs as the eigenfunctions of a Sturm–Liouville problem, and show that they are the eigenfunctions of a compact integral operator as well. This new family extends the PSWFs of order zero to real order $\alpha > -1$, and also generalizes the Gegenbauer polynomials to an orthogonal system furnished with a tuning parameter c. We describe some efficient algorithms for the evaluation of the GPSWFs and the associated eigenvalues, and present some analytic and asymptotic formulas.

3.1. GPSWFs as the eigenfunctions of a Sturm–Liouville operator

Motivated by the two observations made in the introductory section, we define the second-order differential operator

$$\mathcal{D}_x = -\left(1-x^2\right)^{-\alpha}\partial_x\left(\left(1-x^2\right)^{\alpha+1}\partial_x\right) + c^2x^2 = -\left(1-x^2\right)\partial_x^2 + 2(\alpha+1)x\partial_x + c^2x^2, \tag{3.1}$$

where $\alpha > -1$, $c > 0$ and $x \in I$. It is clear that $\mathcal{D}_x$ is a strictly positive self-adjoint operator in the sense that for any u and v in the domain of $\mathcal{D}_x$,

$$(\mathcal{D}_x u, v)_{\omega_\alpha} = (u, \mathcal{D}_x v)_{\omega_\alpha}, \qquad (\mathcal{D}_x u, u)_{\omega_\alpha} = \|\partial_x u\|^2_{\omega_{\alpha+1}} + c^2\|xu\|^2_{\omega_\alpha} > 0, \quad \forall u \neq 0. \tag{3.2}$$

Hence, by the Sturm–Liouville theory (cf. [3,15]), there exists a countable and infinite set of bounded, analytic functions, denoted by $\{\psi_n^{(\alpha)}(x;c)\}_{n=0}^\infty$, satisfying

$$\mathcal{D}_x\psi_n^{(\alpha)}(x;c) = \chi_n^{(\alpha)}\psi_n^{(\alpha)}(x;c), \quad n \geqslant 0,\ x \in I, \tag{3.3}$$

where $\{\chi_n^{(\alpha)} := \chi_n^{(\alpha)}(c)\}_{n=0}^\infty$ are the corresponding eigenvalues. We define the eigenfunction $\psi_n^{(\alpha)}(x;c)$ as the *generalized prolate spheroidal wave function of order* α and of degree n. It is obvious that the GPSWFs include the PSWFs as a special case. Moreover, if $c = 0$, the eigen-problem (3.3) is reduced to (2.1), so we have

$$\psi_n^{(\alpha)}(x;0) = G_n^{(\alpha)}(x), \qquad \chi_n^{(\alpha)}(0) = \gamma_n^{(\alpha)} = n(n+2\alpha+1). \tag{3.4}$$

Accordingly, the GPSWFs can be regarded as a generalization of the Gegenbauer polynomials with a tuning parameter c. The GPSWFs oscillate more uniformly (cf. Fig. 3.1), and the presence of the parameter c is a key advantage of the GPSWFs over the Gegenbauer polynomials.

We summarize below some basic properties of the GPSWFs derived from the Sturm–Liouville theory (cf. [3,15]).

Theorem 3.1. *For any $c > 0$ and $\alpha > -1$,*

(i) $\{\psi_n^{(\alpha)}(x;c)\}_{n=0}^\infty$ *are all real, smooth, and form a complete orthonormal system of* $L^2_{\omega_\alpha}(I)$, *namely,*

$$\int_{-1}^{1}\psi_m^{(\alpha)}(x;c)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx = \delta_{mn}. \tag{3.5}$$

(ii) $\{\chi_n^{(\alpha)}(c)\}_{n=0}^\infty$ *are all real, positive, simple and ordered as*

$$0 < \chi_0^{(\alpha)}(c) < \chi_1^{(\alpha)}(c) < \cdots < \chi_n^{(\alpha)}(c) < \cdots. \tag{3.6}$$

(iii) $\{\psi_n^{(\alpha)}(x;c)\}_{n=0}^\infty$ *with even n are even functions of x, and those with odd n are odd, namely,*

$$\psi_n^{(\alpha)}(-x;c) = (-1)^n\psi_n^{(\alpha)}(x;c), \quad \forall x \in [-1,1]. \tag{3.7}$$

(iv) *$\psi_n^{(\alpha)}(x;c)$ has exactly n real distinct zeros in the interval $(-1,1)$ and between two consecutive zeros of $\psi_{n+1}^{(\alpha)}(x;c)$ there exists exactly one zero of $\psi_n^{(\alpha)}(x;c)$.*

We have the following explicit bounds for the eigenvalues $\{\chi_n^{(\alpha)}(c)\}$, whose proof is given in Appendix B.

Lemma 3.1. *For any $c>0$ and $\alpha>-1$,*

$$n(n+2\alpha+1)<\chi_n^{(\alpha)}(c)<n(n+2\alpha+1)+c^2,\quad n\geqslant 0. \tag{3.8}$$

For $0<c\ll 1$, the GPSWF $\psi_n^{(\alpha)}(x;c)$ is a perturbation of the Gegenbauer polynomial $G_n^{(\alpha)}(x)$.

Lemma 3.2. *For $0<c\ll 1$,*

$$\psi_n^{(\alpha)}(x;c)=G_n^{(\alpha)}(x)+O\big(c^2\big),\qquad \chi_n^{(\alpha)}(c)=\gamma_n^{(\alpha)}+O\big(c^2\big),\quad n\geqslant 0. \tag{3.9}$$

The estimates follow from a perturbation scheme as in [44]. We sketch the proof in Appendix C.

3.2. GPSWFs as the eigenfunctions of an integral operator

A remarkable property of the Slepian functions is that they are the eigenfunctions of both a Sturm–Liouville operator and a compact integral operator (cf. (1.1)–(1.2)). Notably, the GPSWFs inherit such a coincidence.

Define the integral operator $\mathcal{F}_c^{(\alpha)}:L^2_{\omega_\alpha}(I)\to L^2_{\omega_\alpha}(I)$ by

$$\mathcal{F}_c^{(\alpha)}[\phi](x)=\int_{-1}^{1}e^{\mathrm{i}cxt}\phi(t)\omega_\alpha(t)\,dt,\quad x\in(-1,1),\ c>0. \tag{3.10}$$

Obviously, it is compact. Before we show $\mathcal{F}_c^{(\alpha)}$ commutes with the differential operator (3.1), we introduce an associated integral operator $\mathcal{Q}_c^{(\alpha)}:L^2_{\omega_\alpha}(I)\to L^2_{\omega_\alpha}(I)$, defined by

$$\mathcal{Q}_c^{(\alpha)}[\phi](x)=\int_{-1}^{1}\mathcal{K}_c^{(\alpha)}(x,t)\phi(t)\omega_\alpha(t)\,dt,\quad x\in(-1,1),\ c>0, \tag{3.11}$$

where the kernel

$$\mathcal{K}_c^{(\alpha)}(x,t):=\frac{J_{\alpha+\frac12}(c|t-x|)}{(c|t-x|)^{\alpha+\frac12}}, \tag{3.12}$$

and $J_{\alpha+\frac12}(\cdot)$ is the Bessel function of the first kind. We now verify the relation:

$$\mathcal{Q}_c^{(\alpha)}=\frac{1}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}\big(\mathcal{F}_c^{(\alpha)}\big)^*\circ\mathcal{F}_c^{(\alpha)}. \tag{3.13}$$

Indeed, by (3.10),

$$\Big(\big(\mathcal{F}_c^{(\alpha)}\big)^*\circ\mathcal{F}_c^{(\alpha)}\Big)[\phi](x)=\int_{-1}^{1}\left(\int_{-1}^{1}e^{\mathrm{i}c(t-x)s}\omega_\alpha(s)\,ds\right)\phi(t)\omega_\alpha(t)\,dt. \tag{3.14}$$

Observe that the integral inside the brackets remains the same when $t-x$ is in place of $x-t$. Thus, taking $\nu=\alpha+\frac12$ and $z=c|t-x|$ in (2.12) leads to

$$\int_{-1}^{1}e^{\mathrm{i}c(t-x)s}\omega_\alpha(s)\,ds=2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)\frac{J_{\alpha+\frac12}(c|t-x|)}{(c|t-x|)^{\alpha+\frac12}}=2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)\mathcal{K}_c^{(\alpha)}(x,t).$$

Inserting it into (3.14), we derive (3.13) from the definition (3.11).

The following theorem states that the GPSWFs are the eigenfunctions of $\mathcal{F}_c^{(\alpha)}$ and $\mathcal{Q}_c^{(\alpha)}$.

Theorem 3.2. *For any $c > 0$, the GPSWFs are the eigenfunctions of $\mathcal{F}_c^{(\alpha)}$:*

$$\mathcal{F}_c^{(\alpha)}\big[\psi_n^{(\alpha)}\big](x;c) = \mathrm{i}^n \lambda_n^{(\alpha)} \psi_n^{(\alpha)}(x;c), \quad x \in (-1,1),\ \alpha > -1, \tag{3.15}$$

and the eigenvalues $\lambda_n^{(\alpha)} := \lambda_n^{(\alpha)}(c)$ (modulo the factor i^n) are all real and

$$\lambda_0^{(\alpha)} > \lambda_1^{(\alpha)} > \cdots > \lambda_n^{(\alpha)} > \cdots > 0. \tag{3.16}$$

Moreover, $\{\psi_n^{(\alpha)}(x;c)\}_{n=0}^{\infty}$ are also the eigenfunctions of $\mathcal{Q}_c^{(\alpha)}$:

$$\mathcal{Q}_c^{(\alpha)}\big[\psi_n^{(\alpha)}\big](x;c) = \mu_n^{(\alpha)} \psi_n^{(\alpha)}(x;c), \tag{3.17}$$

and the eigenvalues satisfy

$$\mu_n^{(\alpha)} = \frac{1}{2^\alpha \sqrt{2\pi}\,\Gamma(\alpha+1)} \big(\lambda_n^{(\alpha)}\big)^2. \tag{3.18}$$

Proof. Since (3.17)–(3.18) follow from (3.13) and (3.15) directly, it suffices to prove (3.15) and (3.16).

We first prove (3.15) and show that $\mathcal{D}_x$ in (3.1) commutes with $\mathcal{F}_c^{(\alpha)}$. One verifies readily that $\mathcal{D}_x e^{\mathrm{i}ctx} = \mathcal{D}_t e^{\mathrm{i}ctx}$, and

$$\begin{aligned}\chi_n^{(\alpha)} \int_{-1}^{1} e^{\mathrm{i}ctx} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt &\overset{(3.3)}{=} \int_{-1}^{1} \omega_\alpha(t) e^{\mathrm{i}ctx} \mathcal{D}_t \psi_n^{(\alpha)}(t;c)\,dt \overset{(3.2)}{=} \int_{-1}^{1} \omega_\alpha(t)\psi_n^{(\alpha)}(t;c)\mathcal{D}_t e^{\mathrm{i}ctx}\,dt \\ &= \int_{-1}^{1} \omega_\alpha(t)\psi_n^{(\alpha)}(t;c)\mathcal{D}_x e^{\mathrm{i}ctx}\,dt = \mathcal{D}_x \int_{-1}^{1} e^{\mathrm{i}ctx}\psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt,\end{aligned}$$

which implies that $\mathcal{F}_c^{(\alpha)}[\psi_n^{(\alpha)}]$ is an eigenfunction of $\mathcal{D}_x$ with the corresponding eigenvalue $\chi_n^{(\alpha)}$, so it must be proportional to $\psi_n^{(\alpha)}$. We denote the proportional constant by $\mathrm{i}^n\lambda_n^{(\alpha)}$, so (3.15) follows.

Now, we show that $\{\lambda_n^{(\alpha)}\}$ are all real. Taking the complex conjugate of (3.15) leads to

$$\begin{aligned}(-1)^n \mathrm{i}^n \big(\lambda_n^{(\alpha)}\big)^* \psi_n^{(\alpha)} &= \int_{-1}^{1} e^{-\mathrm{i}ctx} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt = \int_{-1}^{1} e^{\mathrm{i}ctx}\psi_n^{(\alpha)}(-t;c)\omega_\alpha(t)\,dt \\ &\overset{(3.7)}{=} (-1)^n \int_{-1}^{1} e^{\mathrm{i}ctx}\psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt \overset{(3.15)}{=} (-1)^n \mathrm{i}^n \lambda_n^{(\alpha)} \psi_n^{(\alpha)},\end{aligned}$$

which implies $(\lambda_n^{(\alpha)})^* = \lambda_n^{(\alpha)}$.

To establish (3.16), we first show that the same ordering holds for $\{\mu_n^{(\alpha)}\}$. Indeed, we infer from the operator theory (cf. [16]) that the eigenvalues $\{\mu_n^{(\alpha)}(c)\}$ are non-degenerated and continuously depend on c. Thus, it suffices to prove that

$$\mu_n^{(\alpha)}(c) > \mu_{n+1}^{(\alpha)}(c), \quad n \geqslant 0, \tag{3.19}$$

holds for some $c > 0$. In other words, (3.19) is valid for all c if it is true for certain c. We justify this claim by contradiction. Suppose that there exists a positive $\tilde{c} \neq c$ that violates this ordering, there must exist a c_1 between c and $\tilde{c}$ such that $\mu_n^{(\alpha)}(c_1) = \mu_{n+1}^{(\alpha)}(c_1)$, which contradicts to the fact that $\{\mu_n^{(\alpha)}\}$ are distinct. In view of this, we next prove (3.19) for some sufficiently small c by following the idea for the proof of the case $\alpha = 0$ in [45]. Indeed, differentiating (3.17) with respect to x gives

$$\mu_n^{(\alpha)} \partial_x \psi_n^{(\alpha)}(x;c) = \int_{-1}^{1} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\partial_x \mathcal{K}_c^{(\alpha)}(x,t)\,dt, \tag{3.20}$$

and

$$\mu_{n+1}^{(\alpha)} \partial_x \psi_{n+1}^{(\alpha)}(x;c) = \int_{-1}^{1} \psi_{n+1}^{(\alpha)}(t;c)\omega_\alpha(t)\partial_x \mathcal{K}_c^{(\alpha)}(x,t)\,dt. \tag{3.21}$$

We also obtain directly from (3.12) that

$$\partial_x \mathcal{K}_c^{(\alpha)}(x,t) = -\partial_t \mathcal{K}_c^{(\alpha)}(x,t). \tag{3.22}$$

Multiplying (3.20) by $\psi_{n+1}^{(\alpha)}\omega_\alpha$ and integrating the resulting equation over $(-1,1)$, leads to

$$\begin{aligned}
\mu_n^{(\alpha)} \int_{-1}^{1} \partial_x \psi_n^{(\alpha)} \psi_{n+1}^{(\alpha)} \omega_\alpha \, dx &= \int_{-1}^{1} \left(\int_{-1}^{1} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\partial_x \mathcal{K}_c^{(\alpha)}(x,t)\,dt \right) \psi_{n+1}^{(\alpha)}(x;c)\omega_\alpha(x)\,dx \\
&\overset{(3.22)}{=} -\int_{-1}^{1} \left(\int_{-1}^{1} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\partial_t \mathcal{K}_c^{(\alpha)}(x,t)\,dt \right) \psi_{n+1}^{(\alpha)}(x;c)\omega_\alpha(x)\,dx \\
&= -\int_{-1}^{1} \left(\int_{-1}^{1} \psi_{n+1}^{(\alpha)}(x;c)\omega_\alpha(x)\partial_t \mathcal{K}_c^{(\alpha)}(x,t)\,dt \right) \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt \\
&= -\mu_{n+1}^{(\alpha)} \int_{-1}^{1} \partial_x \psi_{n+1}^{(\alpha)} \psi_n^{(\alpha)} \omega_\alpha \, dx,
\end{aligned} \tag{3.23}$$

where the last equality is obtained by multiplying (3.21) by $\psi_n^{(\alpha)}\omega_\alpha$ and integrating the resulting equation over $(-1,1)$. For $0 < c \ll 1$, we find from (2.2), (2.8) and (3.9) that

$$\begin{aligned}
\int_{-1}^{1} \partial_x \psi_{n+1}^{(\alpha)} \psi_n^{(\alpha)} \omega_\alpha \, dx &= \int_{-1}^{1} \partial_x G_{n+1}^{(\alpha)} G_n^{(\alpha)} \omega_\alpha \, dx + O\left(c^2\right) = \frac{(n+1)k_{n+1}^{(\alpha)}}{k_n^{(\alpha)}} + O\left(c^2\right) \\
&= \sqrt{\frac{(n+1)(2n+2\alpha+1)(2n+2\alpha+3)}{n+2\alpha+1}} + O\left(c^2\right),
\end{aligned}$$

and

$$\int_{-1}^{1} \partial_x \psi_n^{(\alpha)} \psi_{n+1}^{(\alpha)} \omega_\alpha \, dx = \int_{-1}^{1} \partial_x G_n^{(\alpha)} G_{n+1}^{(\alpha)} \omega_\alpha \, dx + O\left(c^2\right) = O\left(c^2\right).$$

Thus, we derive from (3.23) that for $0 < c \ll 1$,

$$\mu_n^{(\alpha)}(c) - \mu_{n+1}^{(\alpha)}(c) = \mu_n^{(\alpha)}(c)\left(1 + \frac{\int_{-1}^{1} \partial_x \psi_n^{(\alpha)} \psi_{n+1}^{(\alpha)} \omega_\alpha \, dx}{\int_{-1}^{1} \psi_n^{(\alpha)} \partial_x \psi_{n+1}^{(\alpha)} \omega_\alpha \, dx}\right) = \mu_n^{(\alpha)}(c)\left(1 + O\left(c^2\right)\right) \overset{(3.18)}{>} 0, \tag{3.24}$$

which yields the ordering (3.19) for all $c > 0$. Thanks to (3.18), we have $|\lambda_n^{(\alpha)}| > |\lambda_{n+1}^{(\alpha)}|$, which, together with the fact $\lambda_n^{(\alpha)} > 0$ (to be shown below), gives (3.16). □

Next, we present some properties and explicit formulas of the eigenfunctions $\{\lambda_n^{(\alpha)}(c)\}$.

Theorem 3.3.

(1) *For any $\alpha > -1$ and $n \geqslant 0$,*

$$\lim_{c \to 0^+} \frac{\lambda_n^{(\alpha)}(c)}{c^n} = \frac{1}{n!(k_n^{(\alpha)})^2} = \frac{\sqrt{\pi}\,\Gamma(n+\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+3/2)\Gamma(2n+2\alpha+1)}, \tag{3.25}$$

where $k_n^{(\alpha)}$ is the leading coefficient of the Gegenbauer polynomial $G_n^{(\alpha)}(x)$ in (2.8).

(2) *$\{\lambda_n^{(\alpha)}(c)\}$ are all strictly positive.*

(3) *$\lambda_n^{(\alpha)}(c)$ satisfies*

$$\frac{\partial \lambda_n^{(\alpha)}(c)}{\partial c} = \frac{\lambda_n^{(\alpha)}(c)}{c} F(c,\alpha), \quad \forall c > 0, \tag{3.26}$$

where

$$F_n(c,\alpha)=\int_{-1}^{1} x\psi_n^{(\alpha)}(x;c)\partial_x\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx. \tag{3.27}$$

(4) $\lambda_n^{(\alpha)}(c)$ *has the explicit expression*:

$$\lambda_n^{(\alpha)}(c)=\frac{\sqrt{\pi}\,\Gamma(n+\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+3/2)\Gamma(2n+2\alpha+1)}\cdot c^n\cdot\exp\left(\int_0^c\frac{F_n(\tau,\alpha)-n}{\tau}\,d\tau\right),\quad \forall c>0. \tag{3.28}$$

(5) *We have*

$$\sum_{n=0}^{\infty}\bigl(\lambda_n^{(\alpha)}(c)\bigr)^2=\frac{\pi\,\Gamma^2(\alpha+1)}{\Gamma^2(\alpha+3/2)},\qquad \sum_{n=0}^{\infty}\mu_n^{(\alpha)}(c)=\frac{\pi^{3/2}2^{\alpha+1/2}\Gamma^3(\alpha+1)}{\Gamma^2(\alpha+3/2)}. \tag{3.29}$$

Proof. (1) We first prove (3.25). Differentiating (3.15) n times with respect to x and setting $x=0$ in the resulting equation, leads to

$$\mathrm{i}^n\lambda_n^{(\alpha)}(c)\partial_x^n\psi_n^{(\alpha)}(0;c)=(\mathrm{i}c)^n\int_{-1}^{1}t^n\psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt. \tag{3.30}$$

Thanks to (3.9), we find from (2.2) and (2.8) that

$$\lim_{c\to0^+}\partial_x^n\psi_n^{(\alpha)}(0;c)=n!k_n^{(\alpha)},\qquad \lim_{c\to0^+}\int_{-1}^{1}t^n\psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt=\frac{1}{k_n^{(\alpha)}}.$$

Working out the constants by using (2.8), we derive (3.25) from (3.30).

(2) The formula (3.25) implies that for sufficient small c, $\lambda_n^{(\alpha)}(c)>0$. In fact, this property holds for all $c>0$, since if there exists $\tilde c>0$ such that $\lambda_n^{(\alpha)}(\tilde c)<0$, we can find $c_1>0$ such that $\lambda_n^{(\alpha)}(c_1)=0$, which is not possible.

(3) We now turn to the proof of (3.26). For $0<b<c$, we multiply (3.15) by $\psi_n^{(\alpha)}(x;b)\omega_\alpha(x)$ and integrate the resulting equation over $(-1,1)$ to derive that

$$\begin{aligned}\mathrm{i}^n\lambda_n^{(\alpha)}(c)\int_{-1}^{1}\psi_n^{(\alpha)}(x;b)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx &= \int_{-1}^{1}\psi_n^{(\alpha)}(x;b)\left(\int_{-1}^{1}e^{\mathrm{i}cxt}\psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt\right)\omega_\alpha(x)\,dx\\ &= \int_{-1}^{1}\psi_n^{(\alpha)}(t;c)\left(\int_{-1}^{1}e^{\mathrm{i}cxt}\psi_n^{(\alpha)}(x;b)\omega_\alpha(x)\,dx\right)\omega_\alpha(t)\,dt\\ &= \int_{-1}^{1}\psi_n^{(\alpha)}(t;c)\left(\int_{-1}^{1}e^{\mathrm{i}bx\frac{ct}{b}}\psi_n^{(\alpha)}(x;b)\omega_\alpha(x)\,dx\right)\omega_\alpha(t)\,dt\\ &\overset{(3.15)}{=} \mathrm{i}^n\lambda_n^{(\alpha)}(b)\int_{-1}^{1}\psi_n^{(\alpha)}(t;c)\psi_n^{(\alpha)}\left(\frac{ct}{b};b\right)\omega_\alpha(t)\,dt.\end{aligned}$$

Thus, by using Taylor's expansion,

$$\begin{aligned}&\frac{\lambda_n^{(\alpha)}(c)-\lambda_n^{(\alpha)}(b)}{c-b}\int_{-1}^{1}\psi_n^{(\alpha)}(x;b)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx\\ &=\lambda_n^{(\alpha)}(b)\int_{-1}^{1}\psi_n^{(\alpha)}(x;c)\frac{\psi_n^{(\alpha)}(\frac{cx}{b};b)-\psi_n^{(\alpha)}(x;b)}{c-b}\omega_\alpha(x)\,dx\\ &=\lambda_n^{(\alpha)}(b)\int_{-1}^{1}\psi_n^{(\alpha)}(x;c)\frac{\partial_x\psi_n^{(\alpha)}(x;b)(\frac{cx}{b}-x)+O(\frac{cx}{b}-x)^2}{c-b}\omega_\alpha(x)\,dx\end{aligned}$$

$$= \frac{\lambda_n^{(\alpha)}(b)}{b} \int_{-1}^{1} \psi_n^{(\alpha)}(x;c)\big(\partial_x \psi_n^{(\alpha)}(x;b)x + O\big((c-b)x\big)\big)\omega_\alpha(x)\,dx.$$

Obviously, the above identities also hold when we interchange c and b. Letting $b \to c$ yields

$$\frac{\partial \lambda_n^{(\alpha)}(c)}{\partial c} = \frac{\lambda_n^{(\alpha)}(c)}{c} \int_{-1}^{1} x\psi_n^{(\alpha)}(x;c)\partial_x \psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx. \tag{3.31}$$

(4) Suppose that c_0 and c are two positive real numbers such that $0 < c_0 < c$. In view of $\lambda_n^{(\alpha)}(c) > 0$, integrating (3.26) from c_0 to c yields

$$\begin{aligned}\ln\big(\lambda_n^{(\alpha)}(c)\big) &= \ln\big(\lambda_n^{(\alpha)}(c_0)\big) + \int_{c_0}^{c} \frac{1}{\tau}\left(\int_{-1}^{1} x\psi_n^{(\alpha)}(x;\tau)\partial_x\psi_n^{(\alpha)}(x;\tau)\omega_\alpha(x)\,dx\right)d\tau \\ &= \ln\big(\lambda_n^{(\alpha)}(c_0)\big) + n\ln\frac{c}{c_0} + \int_{c_0}^{c}\left(\frac{1}{\tau}\int_{-1}^{1} x\psi_n^{(\alpha)}(x;\tau)\partial_x\psi_n^{(\alpha)}(x;\tau)\omega_\alpha(x)\,dx - \frac{n}{\tau}\right)d\tau. \end{aligned} \tag{3.32}$$

Exponentiating the above equation gives

$$\lambda_n^{(\alpha)}(c) = c^n \cdot \frac{\lambda_n^{(\alpha)}(c_0)}{c_0^n} \cdot \exp\left(\int_{c_0}^{c} \frac{F_n(\tau,\alpha) - n}{\tau}\,d\tau\right). \tag{3.33}$$

We infer from (2.9) and (3.9) that for any $0 < \tau \ll 1$,

$$F_n(\tau,\alpha) = \int_{-1}^{1} xG_n^{(\alpha)}\partial_x G_n^{(\alpha)}\omega_\alpha\,dx + O\big(\tau^2\big) = n + O\big(\tau^2\big),$$

which implies

$$\lim_{c_0\to 0}\int_{c_0}^{c} \frac{F_n(\tau,\alpha) - n}{\tau}\,d\tau = \int_{0}^{c} \frac{F_n(\tau,\alpha) - n}{\tau}\,d\tau. \tag{3.34}$$

Consequently, the formula (3.28) follows from (3.25) and (3.33)–(3.34).

(5) It remains to prove (3.29). We start with the expansion

$$e^{\mathrm{i}cxt} = \sum_{n=0}^{\infty}\left(\int_{-1}^{1} e^{\mathrm{i}cx\tau}\psi_n^{(\alpha)}(\tau;c)\omega_\alpha(\tau)\,d\tau\right)\psi_n^{(\alpha)}(t;c) \overset{(3.15)}{=} \sum_{n=0}^{\infty} \mathrm{i}^n\lambda_n^{(\alpha)}(c)\psi_n^{(\alpha)}(x;c)\psi_n^{(\alpha)}(t;c). \tag{3.35}$$

Multiplying both sides of (3.35) by their own conjugate and $\omega_\alpha(t)\omega_\alpha(x)$ and integrating the resultant equation over $(-1,1)^2$, we derive from the orthogonality of the GPSWFs that

$$\int_{-1}^{1}\int_{-1}^{1} \big|e^{\mathrm{i}cxt}\big|^2\omega_a(t)\omega_a(x)\,dx\,dt = \left(\int_{-1}^{1}\omega_\alpha(x)\,dx\right)^2 \overset{(2.2)}{=} \frac{1}{(k_0^{(\alpha)})^4} \overset{(2.8)}{=} \frac{\pi\Gamma^2(\alpha+1)}{\Gamma^2(\alpha+3/2)} = \sum_{n=0}^{\infty}\big|\lambda_n^{(\alpha)}(c)\big|^2, \tag{3.36}$$

which yields the first identity in (3.29). The second one can be derived from (3.18) and the first one in (3.29) directly. □

Remark 3.1. The formula (3.28) can be regarded as a generalization of Theorem 9 in [38]. Moreover, it is worthwhile to point out that the summation of the square of all $\{\lambda_n^{(\alpha)}(c)\}$ in (3.29) is a constant independent of c.

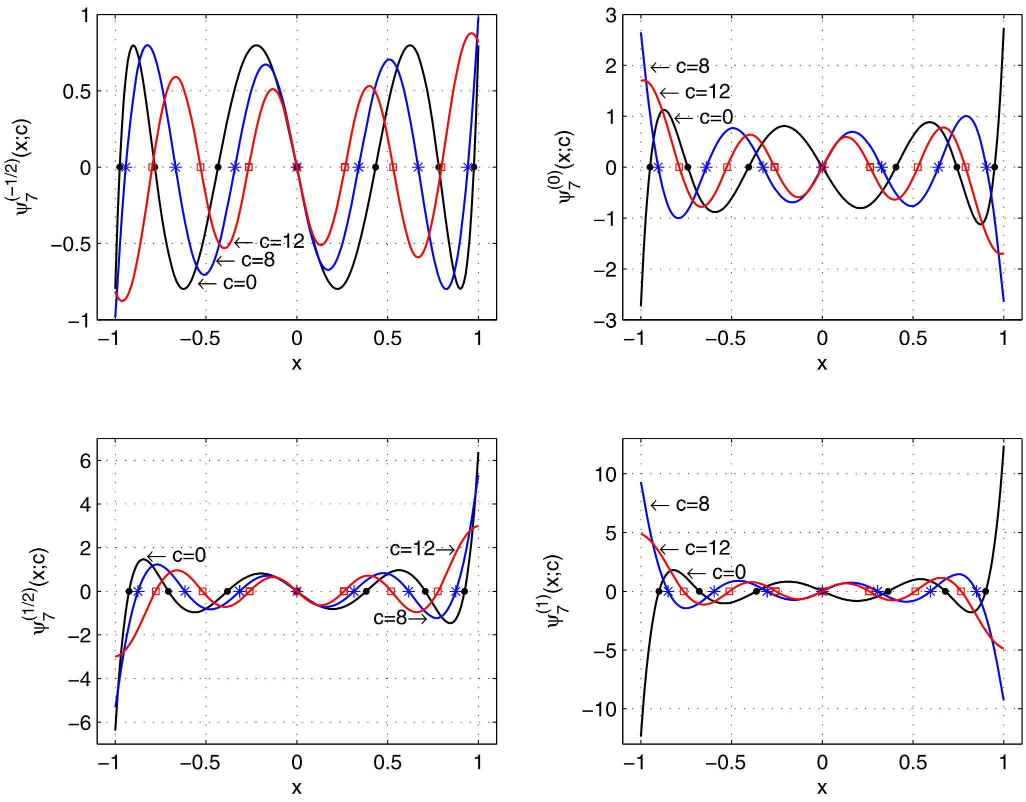

Fig. 3.1. Graphs of $\psi_7^{(\alpha)}(x;c)$ with $\alpha = \pm 1/2, 0, 1$ and $c = 0, 8, 12$.

3.3. Numerical evaluation of the GPSWFs and the eigenvalues

As with the Slepian functions, an efficient approach to evaluate the GPSWFs is the Bouwkamp-type algorithm (cf. [8,57,12]). Basically, we expand $\{\psi_n^{(\alpha)}(x;c)\}$ in terms of the normalized Gegenbauer polynomials:

$$\psi_n^{(\alpha)}(x;c) = \sum_{k=0}^{\infty} \beta_k^n G_k^{(\alpha)}(x) \quad \text{with } \beta_k^n = \int_{-1}^{1} \psi_n^{(\alpha)}(x;c) G_k^{(\alpha)}(x)\omega_\alpha(x)\,dx. \tag{3.37}$$

Substituting this expansion into (3.3) and using the properties (2.1) and (2.3), we obtain an equivalent eigen-problem:

$$\left(\boldsymbol{A} - \chi_n^{(\alpha)} \cdot \boldsymbol{I}\right)\vec{\beta}^n = 0, \tag{3.38}$$

where $\vec{\beta}^n = (\beta_0^n, \beta_1^n, \beta_2^n, \ldots)' \in l^2$ and $\boldsymbol{A}$ is a symmetric five-diagonal matrix with three non-zero diagonals given by

$$A_{k,k} = k(k+2\alpha+1) + d_k \cdot c^2, \qquad A_{k,k+2} = A_{k+2,k} = e_k \cdot c^2, \tag{3.39}$$

where

$$d_k = \frac{2k(k+2\alpha+1)+2\alpha-1}{(2k+2\alpha-1)(2k+2\alpha+3)}, \qquad e_k = \sqrt{\frac{(k+1)(k+2)(k+2\alpha+1)(k+2\alpha+2)}{(2k+2\alpha+1)(2k+2\alpha+3)^2(2k+2\alpha+5)}}. \tag{3.40}$$

The eigen-system (3.38) involves infinitely many unknowns, so an appropriate truncation is needed. For fixed n, the coefficient β_k^n decays exponentially with respect to k (cf. Theorem 3.4 below), so a reasonable truncation could lead to a very accurate computation of the GPSWFs and the associated eigenvalues. Boyd [12] suggested a cut-off: $M = 2N + 30$ for an accurate evaluation of the Slepian functions. Here, we use the truncation $M = 2N + 2\alpha + 30$ for the computations of $\{\psi_n^{(\alpha)}, \chi_n^{(\alpha)}\}_{n=0}^N$. We also notice that $\beta_k^n = 0$ if $n+k$ is odd, which allows us to reduce the system (3.38) (with a truncation) to a symmetric tridiagonal system, and efficient eigen-solvers can be applied.

In Fig. 3.1, we plot some samples of the GPSWFs, and we see that as c increases to some degree, $\psi_n^{(\alpha)}(x)$ oscillates more and more uniformly, so its zeros become quasi-uniformly distributed.

We now turn to the numerical evaluations of $\{\lambda_n^{(\alpha)}(c)\}$, which become exponentially small, when n is large (cf. (3.61) below). Hence, it is necessary to propose a stable algorithm.

Lemma 3.3. *For any $\alpha > -1$ and $c > 0$,*

$$\lambda_n^{(\alpha)}(c) = \begin{cases} \dfrac{\beta_0^n}{\mathrm{i}^n \psi_n^{(\alpha)}(0;c)} \sqrt{\dfrac{\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}}, & \text{if } n \text{ is even}, \\ \dfrac{c\beta_1^n}{\mathrm{i}^{n-1} \partial_x \psi_n^{(\alpha)}(0;c)} \sqrt{\dfrac{\sqrt{\pi}\,\Gamma(\alpha+1)}{(2\alpha+3)\Gamma(\alpha+3/2)}}, & \text{if } n \text{ is odd}, \end{cases} \tag{3.41}$$

where β_0^n and β_1^n are given in (3.37)*.*

Proof. Taking $x=0$ in (3.15) leads to

$$\mathrm{i}^n \lambda_n^{(\alpha)} \psi_n^{(\alpha)}(0;c) = \int_{-1}^{1} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt.$$

The parity of PSWFs and the fact that $\psi_n^{(\alpha)}(x;c)$ has exactly n real zeros in $(-1,1)$, imply that $\psi_n^{(\alpha)}(0;c) \neq 0$ for even n, while $\psi_n^{(\alpha)}(0;c) = 0$ for odd n. Therefore, for even n,

$$\begin{aligned} \lambda_n^{(\alpha)}(c) &= \frac{1}{\mathrm{i}^n \psi_n^{(\alpha)}(0;c)} \int_{-1}^{1} \psi_n^{(\alpha)}(t;c)\omega_\alpha(t)\,dt = \frac{1}{\mathrm{i}^n \psi_n^{(\alpha)}(0;c)} \int_{-1}^{1} \left(\sum_{k=0}^{\infty} \beta_k^n G_k^{(\alpha)}(t) \right) \omega_\alpha(t)\,dt \\ &\overset{(2.2)}{=} \frac{\beta_0^n}{\mathrm{i}^n \psi_n^{(\alpha)}(0;c) k_0^{(\alpha)}} \overset{(2.8)}{=} \frac{\beta_0^n}{\mathrm{i}^n \psi_n^{(\alpha)}(0;c)} \sqrt{\frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}}. \end{aligned}$$

For odd n, differentiating (3.15) with respect to x and taking $x=0$, yields

$$\begin{aligned} \lambda_n^{(\alpha)}(c) &= \frac{c}{\mathrm{i}^{n-1} \partial_x \psi_n^{(\alpha)}(0;c)} \int_{-1}^{1} t\psi_n^{(\alpha)}(t;c)\omega_\alpha\,dt = \frac{c}{\mathrm{i}^{n-1} \partial_x \psi_n^{(\alpha)}(0;c)} \sum_{k=0}^{\infty} \beta_k^n \int_{-1}^{1} tG_k^{(\alpha)}(t)\omega_\alpha(t)\,dt \\ &\overset{(2.2)}{=} \frac{c}{\mathrm{i}^{n-1} \partial_x \psi_n^{(\alpha)}(0;c)} \frac{\beta_1^n}{k_1^{(\alpha)}}. \end{aligned}$$

Hence, working out the constant $k_1^{(\alpha)}$ by using (2.8) leads to (3.41) with odd n. □

Notice that the absolute value of the denominator in (3.41) is bounded away from zero, which can be verified by using (3.60) and the formulas for the values of $G_n^{(\alpha)}(0)$ and $\partial_x G_n^{(\alpha)}(0)$ (cf. [47]). Therefore, the computation based on (3.41) is stable even for large n (cf. Fig. 3.3).

3.4. Asymptotic properties

To this end, we study the asymptotic properties of the GPSWFs and the eigenvalues for large degree n. These results will be very useful for the applications of this new system.

We start with an explicit formula for the expansion coefficients $\{\beta_n^m\}$ in (3.37).

Lemma 3.4. *For any $\alpha > -1$ and $c > 0$, we write*

$$\psi_m^{(\alpha)}(x;c) = \sum_{n=0}^{\infty} \beta_n^m G_n^{(\alpha)}(x) \quad \text{with } \beta_n^m = \int_{-1}^{1} \psi_m^{(\alpha)}(x;c) G_n^{(\alpha)}(x)\omega_\alpha(x)\,dx. \tag{3.42}$$

Then we have

$$\beta_n^m = C_{n,\alpha} \frac{(1+(-1)^{m+n})\mathrm{i}^{m+n}}{\lambda_m^{(\alpha)}(c)} \int_0^1 \frac{J_{n+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}} \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt, \tag{3.43}$$

where the constant

$$C_{n,\alpha} = \sqrt{\frac{\pi(2n+\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+1)}}. \tag{3.44}$$

Proof. Since the pair $\{\lambda_m^{(\alpha)}, \psi_m^{(\alpha)}\}$ satisfies

$$\mathrm{i}^m \lambda_m^{(\alpha)} \psi_m^{(\alpha)}(x;c) = \int_{-1}^{1} e^{\mathrm{i}cxt} \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt,$$

one verifies readily that

$$\mathrm{i}^m \lambda_m^{(\alpha)} \left[\frac{1}{\mathrm{i}c}\frac{d}{dx}\right]^k \psi_m^{(\alpha)}(x;c) = \int_{-1}^{1} t^k e^{\mathrm{i}cxt} \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt. \tag{3.45}$$

Let $G_n^{(\alpha)}(\frac{1}{\mathrm{i}c}\frac{d}{dx})$ be the differential operator generalized by the Gegenbauer polynomial $G_n^{(\alpha)}(\cdot)$ as in Lemma 2.1. In view of (3.45), we have that

$$\mathrm{i}^m \lambda_m^{(\alpha)} G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \psi_m^{(\alpha)}(x;c) = \int_{-1}^{1} e^{\mathrm{i}cxt} G_n^{(\alpha)}(t) \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt. \tag{3.46}$$

Taking $x=0$ in (3.46), we obtain from (3.42) that

$$\beta_n^m = \int_{-1}^{1} G_n^{(\alpha)}(t)\psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt = \mathrm{i}^m \lambda_m^{(\alpha)} G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right)\psi_m^{(\alpha)}(0;c). \tag{3.47}$$

To evaluate the right hand side of the above formula, we use (3.11) and (3.17)–(3.18) to derive that

$$G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right)\psi_m^{(\alpha)}(x;c) = \frac{1}{\mu_m^{(\alpha)}} \int_{-1}^{1} \left\{ G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \mathcal{K}_c^{(\alpha)}(x,t) \right\} \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt, \tag{3.48}$$

where $\mathcal{K}_c^{(\alpha)}$ is given in (3.12). We deduce from Lemma 2.1 that for all $t \leqslant x$,

$$G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \mathcal{K}_c^{(\alpha)}(x,t) = \mathrm{i}^n h_n^{(\alpha)} \frac{J_{n+\alpha+\frac{1}{2}}(c(x-t))}{(c(x-t))^{\alpha+\frac{1}{2}}}. \tag{3.49}$$

Similarly, we have that for all $t > x$,

$$G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \mathcal{K}_c^{(\alpha)}(x,t) = (-1)^n \mathrm{i}^n h_n^{(\alpha)} \frac{J_{n+\alpha+\frac{1}{2}}(c(t-x))}{(c(t-x))^{\alpha+\frac{1}{2}}}. \tag{3.50}$$

Therefore, by (2.15), (3.18), (3.48)–(3.50) and the parity of $\psi_m^{(\alpha)}$,

$$\begin{aligned} G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right)\psi_m^{(\alpha)}(0;c) &= \frac{1}{\mu_m^{(\alpha)}} \left\{ \int_{0}^{1} G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \mathcal{K}_c^{(\alpha)}(0;t)\psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt \right. \\ &\quad \left. + \int_{-1}^{0} G_n^{(\alpha)}\left(\frac{1}{\mathrm{i}c}\frac{d}{dx}\right) \mathcal{K}_c^{(\alpha)}(0;t)\psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt \right\} \\ &= 2^{\alpha+\frac{1}{2}}\sqrt{\pi}\,\Gamma(\alpha+1) h_n^{(\alpha)} \frac{(1+(-1)^{m+n})\mathrm{i}^{m+n}}{|\lambda_m^{(\alpha)}|^2} \int_{0}^{1} \frac{J_{n+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}} \psi_m^{(\alpha)}(t;c)\omega_\alpha(t)\,dt. \end{aligned}$$

Notice that $\lambda_m^{(\alpha)}$ is positive, so substituting it into (3.47) and working out the constant by using (3.18) leads to (3.43). □

With the aid of the above theorem, we are able to analyze the asymptotic behavior of β_n^m for large n and even $m+n$ (note: $\beta_n^m = 0$ for odd $m+n$), and show that β_n^m decays super-geometrically with respect to n.

Theorem 3.4. *Let $\{\beta_n^m\}$ be the coefficients defined in (3.42). If $m \geqslant 0$, $\alpha > -1$ and $c > 0$ are fixed and $m + n$ is even, we have that for sufficiently large n*

$$\left|\beta_n^m\right| \leqslant \frac{C}{\lambda_m^{(\alpha)}(c) \cdot n^{1+\frac{\alpha}{2}}}\left[\frac{ce}{2n+2\alpha+1}\right]^n, \tag{3.51}$$

where C is a positive constant independent of n.

Proof. By the asymptotic formula (2.13),

$$\frac{J_{n+\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}} \leqslant \frac{C_1 c^n}{\sqrt{\pi(2n+2\alpha+1)}}\left[\frac{e}{2n+2\alpha+1}\right]^{n+\alpha+\frac{1}{2}} t^n, \quad \forall n \gg 1,\ t > 0, \tag{3.52}$$

where C_1 is a positive constant independent of n. Hence, by (3.43)–(3.44) and the Cauchy–Schwarz inequality,

$$\left|\beta_n^m\right| \leqslant \frac{2C_1 c^n}{\lambda_m^{(\alpha)}}\sqrt{\frac{(2n+\alpha+1)\Gamma(n+2\alpha+1)}{(2n+2\alpha+1)\Gamma(n+1)}}\left[\frac{e}{2n+2\alpha+1}\right]^{n+\alpha+\frac{1}{2}} \times \left(\int_0^1 t^{2n}\omega_\alpha(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^1 \left(\psi_m^{(\alpha)}(t;c)\right)^2\omega_\alpha(t)\,dt\right)^{\frac{1}{2}}. \tag{3.53}$$

Notice that the last integral is $1/2$, and

$$\int_0^1 t^{2n}\omega_\alpha(t)\,dt = \frac{1}{2}\int_0^1 s^{n-\frac{1}{2}}(1-s)^\alpha\,ds = \frac{1}{2}B\left(n+\frac{1}{2},\alpha+1\right) = \frac{\Gamma(n+\frac{1}{2})\Gamma(\alpha+1)}{2\Gamma(n+\alpha+\frac{3}{2})}, \tag{3.54}$$

where we have used the property of the Beta function:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}, \quad \forall p,q > 0.$$

Finally, using the Stirling's formula:

$$\Gamma(x) = \sqrt{2\pi}x^{x-1/2}e^{-x}\left\{1+\frac{1}{12x}+O\left(x^{-3}\right)\right\}, \quad x \gg 1, \tag{3.55}$$

to obtain the asymptotic estimates of the Gamma function of n, we work out the constants and derive (3.51). □

Remark 3.2. As a direct consequence of Theorem 3.4, we find that for $\alpha = 0$,

$$\left|\beta_n^m\right| \leqslant \frac{C}{\lambda_m^{(0)}(c) \cdot n}\left[\frac{ce}{2n+1}\right]^n, \tag{3.56}$$

which indicates a super-geometric decay. Hence, it improves the estimate in Theorem 3.4 of [57], where a geometric decay $O(2^{1-n})$ was predicted.

Next, we present some asymptotic formulas for the GPSWF $\psi_n^{(\alpha)}(x;c)$ with large n by using the inverse power method as described in Rokhlin and Xiao [38] for the Slepian functions. It is also pointed out in [38] that such a procedure is somehow heuristic and the rigorous proof is lengthy and elementary. Here, we sketch this technique and provide some numerical illustrations for the resultant formulas.

Let $\boldsymbol{A}^{(n)} = (A_{ij})_{n-4\leqslant i,j\leqslant n+4}$ be a 9×9 sub-matrix of $\boldsymbol{A}^{(n)}$ given in (3.38)–(3.40). Note that $\beta_{n\pm1}^n = \beta_{n\pm3}^n = 0$, so we are able to reduce $\boldsymbol{A}^{(n)}$ to a 5×5 symmetric tridiagonal matrix $\boldsymbol{B}^{(n)}$ whose main diagonal is

$$(A_{n-4,n-4}, A_{n-2,n-2}, A_{n,n}, A_{n+2,n+2}, A_{n+4,n+4}),$$

and upper off-diagonal is

$$(A_{n-4,n-2}, A_{n-2,n}, A_{n,n+2}, A_{n+2,n+4})$$

(refer to (3.39)–(3.40) for the expressions of A_{ij}). Using the standard inverse power method as an analytic tool, we are able to find a good approximation of the eigenvectors and eigenvalue of $\boldsymbol{B}^{(n)}$:

Table 3.1
Error $E_n^{(\alpha)}(c)$ with $\alpha = -0.5, 0, 0.5, 2$.

c	n	$E_n^{(-1/2)}$	$E_n^{(0)}$	$E_n^{(1/2)}$	$E_n^{(2)}$
1	10	1.56E−05	1.16E−06	3.93E−05	2.06E−04
2	10	6.18E−05	6.97E−06	1.60E−04	8.26E−04
5	10	3.67E−04	1.32E−04	1.13E−03	5.27E−03
1	20	9.76E−07	4.20E−08	2.68E−06	1.68E−05
5	20	2.39E−05	5.60E−06	7.45E−05	4.32E−04
10	20	9.31E−05	7.11E−05	3.82E−04	1.87E−03
1	100	1.56E−09	1.52E−11	4.60E−09	3.49E−08
10	100	1.56E−07	3.10E−08	5.17E−07	3.62E−06
50	100	5.76E−06	1.46E−05	4.16E−05	1.60E−04
50	200	2.48E−07	5.71E−07	1.85E−06	8.36E−06
50	300	4.81E−08	7.80E−08	2.98E−07	1.50E−06
50	400	1.52E−08	1.87E−08	8.26E−08	4.50E−07
50	500	6.23E−09	6.17E−09	3.09E−08	1.77E−07

$$\left(\tilde\beta^n_{n-4}, \tilde\beta^n_{n-2}, \tilde\beta^n_{n}, \tilde\beta^n_{n+2}, \tilde\beta^n_{n+4}\right)^T \simeq \left(\beta^n_{n-4}, \beta^n_{n-2}, \beta^n_{n}, \beta^n_{n+2}, \beta^n_{n+4}\right)^T, \qquad \tilde\chi_n^{(\alpha)}(c) \simeq \chi_n^{(\alpha)}(c),$$

for large n.

The asymptotic formulas are presented below.

Proposition 3.1. *For fixed $c > 0$, $\alpha > -1$ and large n, we have*

$$\beta^n_{n\pm 2k} = \tilde\beta^n_{n\pm 2k} + O\left(\frac{c^2}{n^3}\right), \quad k = 0, 1, 2, \qquad \chi_n^{(\alpha)}(c) = \tilde\chi_n^{(\alpha)}(c) + O\left(\frac{c^2}{n^3}\right), \tag{3.57}$$

where

$$\begin{aligned} &\tilde\beta^n_{n-4} = \tilde\beta^n_{n+4} = \frac{c^4}{512n^2}, \qquad \tilde\beta^n_{n-2} = \frac{c^2}{16n} + (1-2\alpha)\frac{c^2}{32n^2}, \\ &\tilde\beta^n_{n} = 1 - \frac{c^4}{256n^2}, \qquad \tilde\beta^n_{n+2} = -\frac{c^2}{16n} + (3+2\alpha)\frac{c^2}{32n^2}, \end{aligned} \tag{3.58}$$

and

$$\tilde\chi_n^{(\alpha)}(c) = n(n+2\alpha+1) + \frac{c^2}{2} + \frac{c^2(c^2 + 4(\alpha^2 + 5\alpha + 1))}{32n^2}. \tag{3.59}$$

Remark 3.3. In theory, the inverse power method can be used to find the formulas for $\beta^n_{n\pm 2k}$ for $k \geqslant 3$ (see Theorems 10 and 11 in [38] for the case $\alpha = 0$ with $k = 7$). However, the calculations will be very tedious.

As a numerical illustration, we tabulate in Table 3.1 the error $E_n^{(\alpha)}(c) = |1 - \chi_n^{(\alpha)}(c)/\tilde\chi_n^{(\alpha)}(c)|$ for various α, c and n, which indicates that the asymptotic estimate $\tilde\chi_n^{(\alpha)}(c)$ provides a good approximation to $\chi_n^{(\alpha)}(c)$ even for small n. Moreover, we plot in Fig. 3.2 (left), the error $\log_{10}(\max_{0\leqslant|k|\leqslant 2} |(\beta^n_{n+2k} - \tilde\beta^n_{n+2k})/c^2|)$ against $\log_{10}(n)$ for various α and c. We observe that slopes of the lines are approximately -3, which agree with (3.57).

An immediate consequence of Lemma 3.1 is the following important asymptotic formula.

Corollary 3.1. *For fixed $c > 0$, $\alpha > -1$ and large n,*

$$\psi_n^{(\alpha)}(x; c) \simeq \sum_{k=-2}^{2} \beta^n_{n+2k} G^{(\alpha)}_{n+2k}(x), \tag{3.60}$$

where $\{\beta^n_{n\pm 2k}\}_{k=0}^2$ are given by (3.57)–(3.58)*.*

With the aid of the formula (3.28) and the asymptotic estimate (3.60), we are able to analyze the asymptotic behavior of $\lambda_n^{(\alpha)}(c)$ for large n.

Corollary 3.2. *For fixed $c > 0$ and $\alpha > -1$, $\lambda_n^{(\alpha)}(c)$ decays super-geometrically with respect to n:*

$$\lambda_n^{(\alpha)}(c) \simeq \nu_n^{(\alpha)}(c) := c^n \frac{\sqrt{\pi}\,\Gamma(n+\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+3/2)\Gamma(2n+2\alpha+1)} \simeq \frac{e^\alpha}{4^\alpha}\sqrt{\frac{\pi e}{2n+2\alpha+3}}\left[\frac{ce}{4n+4\alpha+2}\right]^n, \quad n \gg 1. \tag{3.61}$$

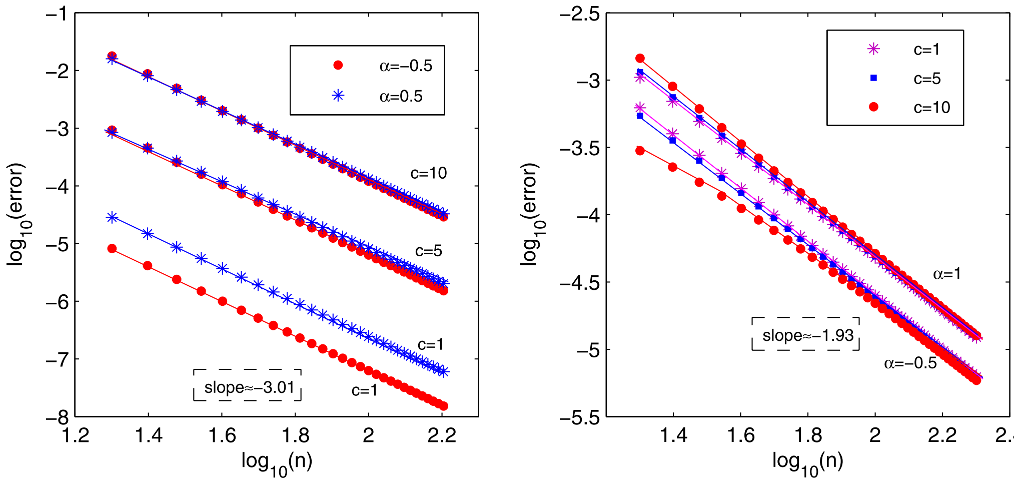

Fig. 3.2. Left: $\log_{10}(\max_{0\leqslant|k|\leqslant 2}|(\beta^n_{n+2k}-\tilde\beta^n_{n+2k})/c^2|)$ for various $n\in[20,160]$, $c=1,5,10$ and $\alpha=-0.5,0.5$. Right: $\log_{10}(|(F_n(c,\alpha)-n)/c^2|)$ for various $n\in[20,200]$, $c=1,5,10$ and $\alpha=-0.5,1$.

Proof. We first show that for fixed $c>0$ and $\alpha>-1$,

$$F_n(c,\alpha)=\int_{-1}^{1} x\psi_n^{(\alpha)}\partial_x\psi_n^{(\alpha)}\omega_\alpha\,dx=n+c^2\cdot O\left(\frac{1}{n^2}\right). \tag{3.62}$$

We derive from the orthogonality (2.2) and (3.60) that

$$F_n(c,\alpha)\simeq\sum_{k=-2}^{2}\sum_{l=k}^{2}\left(\beta^n_{n+2k}\beta^n_{n+2l}\int_{-1}^{1} xG^{(\alpha)}_{n+2k}\partial_x G^{(\alpha)}_{n+2l}\omega_\alpha\,dx\right), \tag{3.63}$$

and verify by using (2.2), (2.5)–(2.6) and (2.8) that

$$\begin{aligned}
&\int_{-1}^{1} xG^{(\alpha)}_{n+2k}\partial_x G^{(\alpha)}_{n+2k}\omega_\alpha\,dx=n+2k,\quad k=0,\pm1,\pm2,\\
&\int_{-1}^{1} xG^{(\alpha)}_{n+2k}\partial_x G^{(\alpha)}_{n+2k+2}\omega_\alpha\,dx=g^{(\alpha)}_{n+2k+2},\quad k=-2,-1,0,1,\\
&\int_{-1}^{1} xG^{(\alpha)}_{n-4}\partial_x G^{(\alpha)}_{n}\omega_\alpha\,dx=\tilde b_{n-1}g^{(\alpha)}_{n-2},\qquad \int_{-1}^{1} xG^{(\alpha)}_{n-2}\partial_x G^{(\alpha)}_{n+2}\omega_\alpha\,dx=\tilde b_{n+1}g^{(\alpha)}_{n},\\
&\int_{-1}^{1} xG^{(\alpha)}_{n}\partial_x G^{(\alpha)}_{n+4}\omega_\alpha\,dx=\tilde b_{n+3}g^{(\alpha)}_{n+2},\qquad \int_{-1}^{1} xG^{(\alpha)}_{n-4}\partial_x G^{(\alpha)}_{n+2}\omega_\alpha\,dx=\tilde b_{n+1}\tilde b_{n-1}g^{(\alpha)}_{n-2},\\
&\int_{-1}^{1} xG^{(\alpha)}_{n-4}\partial_x G^{(\alpha)}_{n+4}\omega_\alpha\,dx=\tilde b_{n+3}\tilde b_{n+1}\tilde b_{n-1}g^{(\alpha)}_{n-2},\qquad \int_{-1}^{1} xG^{(\alpha)}_{n-2}\partial_x G^{(\alpha)}_{n+4}\omega_\alpha\,dx=\tilde b_{n+3}\tilde b_{n+1}g^{(\alpha)}_{n},
\end{aligned} \tag{3.64}$$

where we denoted by

$$g_n^{(\alpha)}=\tilde a_{n-1}\frac{k^{(\alpha)}_{n-2}}{k^{(\alpha)}_{n-1}}+(n-2)\tilde b_{n-1}=\sqrt{\frac{n(n-1)(2n+2\alpha-3)(2n+2\alpha+1)}{(n+2\alpha-1)(n+2\alpha)}}.$$

It is clear that

$$g_n^{(\alpha)}=2n-(2\alpha+1)+\frac{1-4\alpha^2}{n}+O\left(n^{-2}\right).$$

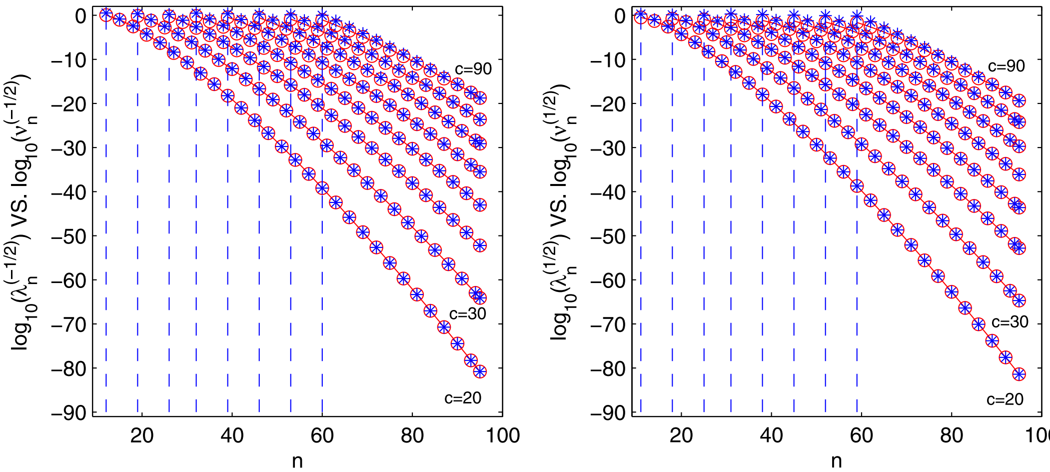

Fig. 3.3. Graphs of $\log_{10}(\lambda_n^{(\alpha)}(c))$ (marked by "∘") vs $\log_{10}(\nu^{(\alpha)}(c))$ (marked by "∗") for various n and $c = 20, 30, \ldots, 90$. Left: $\alpha = -0.5$ and right: $\alpha = 0.5$.

Hence, the summation in (3.63) involves the following terms, which can be computed by using (3.57)–(3.58) as follows:

$$n(\beta_n^n)^2 = n - \frac{c^4}{128n} + c^8 \cdot O(n^{-3}), \qquad (n-2)(\beta_{n-2}^n)^2 = \frac{c^4}{256n} + c^4 \cdot O(n^{-2}),$$
$$(n+2)(\beta_{n+2}^n)^2 = \frac{c^4}{256n} + c^4 \cdot O(n^{-2}), \qquad \beta_{n-2}^n \beta_n^n g_n^{(\alpha)} = (1+2\alpha)\frac{c^2}{16n} + c^2 \cdot O(n^{-2}),$$
$$\beta_n^n \beta_{n+2}^n g_{n+2}^{(\alpha)} = -(1+2\alpha)\frac{c^2}{16n} + c^2 \cdot O(n^{-2}), \qquad \beta_{n-2}^n \beta_{n+2}^n \tilde{b}_{n+1} g_n^{(\alpha)} = -\frac{c^4}{128n} + c^4 \cdot O(n^{-2}),$$
$$\beta_{n-4}^n \beta_n^n \tilde{b}_{n-1} g_{n-2}^{(\alpha)} = \frac{c^4}{256n} + c^8 \cdot O(n^{-3}), \qquad \beta_n^n \beta_{n+4}^n \tilde{b}_{n+3} g_{n+2}^{(\alpha)} = \frac{c^4}{256n} + c^8 \cdot O(n^{-3}),$$

and all the remaining ones

$$(n \pm 4)(\beta_{n\pm4}^n)^2, \qquad \beta_{n-4}^n \beta_{n-2}^n g_{n-2}^{(\alpha)}, \qquad \beta_{n-4}^n \beta_{n+2}^n \tilde{b}_{n+1} \tilde{b}_{n-1} g_{n-2}^{(\alpha)},$$
$$\beta_{n-4}^n \beta_{n+4}^n \tilde{b}_{n+3} \tilde{b}_{n+1} \tilde{b}_{n-1} g_{n-2}^{(\alpha)}, \qquad \beta_{n-2}^n \beta_{n+4}^n \tilde{b}_{n+3} \tilde{b}_{n+1} g_n^{(\alpha)}, \qquad \beta_{n+2}^n \beta_{n+4}^n g_{n+4}^{(\alpha)}$$

are of order $c^6 \cdot O(n^{-3})$. Summing up all the above terms leads to (3.62).

Therefore, the exponential function in (3.28) has the estimate

$$\exp\left(\int_0^c \frac{F_n(\tau, \alpha) - n}{\tau} \, d\tau \right) \simeq 1, \quad \forall n \gg 1,$$

and (3.28) gives the asymptotic formula:

$$\lambda_n^{(\alpha)}(c) \simeq \nu_n^{(\alpha)}(c) := c^n \frac{\sqrt{\pi}\,\Gamma(n+\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+3/2)\Gamma(2n+2\alpha+1)}, \quad \forall n \gg 1. \tag{3.65}$$

We further obtain from Stirling's formula (3.55) that for fixed $c > 0$ and $\alpha > -1$,

$$\nu_n^{(\alpha)}(c) \simeq \frac{e^\alpha}{4^\alpha} \sqrt{\frac{\pi e}{2n+2\alpha+3}} \left[\frac{ce}{4n+4\alpha+2} \right]^n, \quad \forall n \gg 1. \tag{3.66}$$

This completes the proof. □

To illustrate (3.61) numerically, we plot in Fig. 3.3 $\log_{10}(\lambda_n^{(\alpha)}(c))$ (marked by "∘") against $\log_{10}(\nu_n^{(\alpha)}(c))$ (marked by "∗") for $\alpha = \pm 0.5$ and various c, and the dashed lines are $4n + 4\alpha + 2 = ce$. We see that $\lambda_n^{(\alpha)}(c)$ decays exponentially, and $\nu_n^{(\alpha)}(c)$ fits $\lambda_n^{(\alpha)}(c)$ very well even for $4n + 4\alpha + 2 \approx ce$.

4. Spectral approximation by GPSWFs

In this section, we analyze the approximations of functions by GPSWF series in weighted Sobolev spaces. We construct spectral methods using the GPSWFs as basis functions, and provide some guideline on the choice of c to obtain quasi-uniform grids and achieve spectral accuracy.

4.1. Estimates for the truncation errors

For any $N \in \mathbb{N}$ (the set of non-negative integers), we define the finite-dimensional approximation space

$$X_{N,c}^{(\alpha)} = \operatorname{span}\{\psi_n^{(\alpha)}(x;c):\ 0 \leqslant n \leqslant N\}, \tag{4.1}$$

and consider the $L^2_{\omega_\alpha}$-orthogonal projection $\pi_{N,c}^{(\alpha)} : L^2_{\omega_\alpha}(I) \to X_{N,c}^{(\alpha)}$, defined by

$$\big(\pi_{N,c}^{(\alpha)} u - u, v_N\big)_{\omega_\alpha} = 0, \quad \forall v_N \in X_{N,c}^{(\alpha)}. \tag{4.2}$$

It is clear that

$$\big(\pi_{N,c}^{(\alpha)} u\big)(x) = \sum_{n=0}^{N} \hat{u}_n^{(\alpha)} \psi_n^{(\alpha)}(x;c) \quad \text{with } \hat{u}_n^{(\alpha)} := \hat{u}_n^{(\alpha)}(c) = \int_{-1}^{1} u(x)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx. \tag{4.3}$$

Estimates for approximation of bandlimited functions by the Slepian functions have been carried out in, e.g., [38,55,53]. The analysis of approximability of general non-periodic functions by the Slepian functions in Sobolev spaces was first performed in [14], and the proof of the main result was based on a delicate estimate for the decay property of the coefficients $\{\beta_k^n\}$ in (3.42) with $\alpha = 0$. We recall the estimate in [14]:

$$\big|\hat{u}_N^{(0)}\big| \leqslant C\big(N^{-\frac{2}{3}s}\|u\|_{H^s(I)} + (q_N)^{\delta N}\|u\|_{L^2(I)}\big) \quad \text{with } q_N = \sqrt{c^2/\chi_N^{(0)}}, \tag{4.4}$$

where C and δ are positive constants independent of c, N and u. It indicates that the spectral accuracy can be achieved when $q_N < 1$. However, this estimate is suboptimal (at least for small and modest c), since a convergence rate $O(N^{-s})$ is anticipated. Hereafter, we take a different approach for the analysis of GPSWF approximations, which leads to optimal estimates with a more concise derivation.

We first notice that given a "bandlimited" function

$$u(x) = \int_{-1}^{1} e^{\mathrm{i}cxt}\phi(t)\omega_\alpha(t)\,dt \quad \text{where } \phi \in L^2_{\omega_\alpha}(I). \tag{4.5}$$

One verifies readily from (3.15) and (4.5) that

$$\hat{u}_n^{(\alpha)} = \mathrm{i}^n \lambda_n^{(\alpha)} \hat{\phi}_n^{(\alpha)} \quad \text{with } \hat{\phi}_n^{(\alpha)} = \int_{-1}^{1} \phi(x)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx.$$

Hence, by (3.65)–(3.66),

$$\big|\hat{u}_N^{(\alpha)}\big| \leqslant \lambda_N^{(\alpha)}\big|\hat{\phi}_N^{(\alpha)}\big| \leqslant \frac{e^\alpha}{4^\alpha}\sqrt{\frac{\pi e}{2N+2\alpha+3}}\left[\frac{ce}{4N+4\alpha+2}\right]^N \|\phi\|_{\omega_\alpha}, \tag{4.6}$$

which predicts a super-geometric convergence rate when $c < (4N + 4\alpha + 2)/e$.

Our main concern is with the GPSWF approximation of functions in weighted Sobolev spaces. As in [53], we first introduce a Hilbert space associated with the Sturm–Liouville operator $\mathcal{D}_x$ in (3.1). More precisely, define the bilinear form

$$\mathcal{A}_c^{(\alpha)}(\phi, \psi) = \big(\phi', \psi'\big)_{\omega_{\alpha+1}} + c^2(x\phi, x\psi)_{\omega_\alpha}. \tag{4.7}$$

In view of $\mathcal{D}_x\psi_n^{(\alpha)} = \chi_n^{(\alpha)}\psi_n^{(\alpha)}$, the orthogonality follows

$$\mathcal{A}_c^{(\alpha)}\big(\psi_n^{(\alpha)}, \psi_m^{(\alpha)}\big) = \big(\mathcal{D}_x\psi_n^{(\alpha)}, \psi_m^{(\alpha)}\big) = \chi_n^{(\alpha)}\big(\psi_n^{(\alpha)}, \psi_m^{(\alpha)}\big)_{\omega_\alpha} = \chi_n^{(\alpha)}\delta_{mn}. \tag{4.8}$$

Since $\mathcal{D}_x$ is a compact, symmetric and (strictly) positive self-adjoint operator, the fractional power $\mathcal{D}_x^{1/2}$ is well defined, and can be characterized by (cf. [49]):

$$\big\|\mathcal{D}_x^{1/2}u\big\|_{\omega_\alpha}^2 = \mathcal{A}_c^{(\alpha)}(u,u) \quad \Rightarrow \quad \big\|\mathcal{D}_x^{m+1/2}u\big\|_{\omega_\alpha} = \mathcal{A}_c^{(\alpha)}\big(\mathcal{D}_x^m u, \mathcal{D}_x^m u\big), \quad \forall m \in \mathbb{N}. \tag{4.9}$$

For any integer $r \geqslant 0$, we introduce the Hilbert space:

$$\tilde{H}^r_{\omega_\alpha,c}(I) = \left\{u \in L^2_{\omega_\alpha}(I)\colon\ \|u\|_{\tilde{H}^r_{\omega_\alpha,c}} = \left\|\mathcal{D}_x^{r/2}u\right\|_{\omega_\alpha} = \sqrt{\left(\mathcal{D}_x^{r/2}u, \mathcal{D}_x^{r/2}u\right)_{\omega_\alpha}} < \infty\right\}, \tag{4.10}$$

while for real $r > 0$, $\tilde{H}^r_{\omega_\alpha,c}(I)$ is defined by space interpolation as in [2]. Formally, we derive from the orthogonality (3.5) and (4.8) that for any $m \in \mathbb{N}$,

$$\left\|\mathcal{D}_x^m u\right\|^2_{\omega_\alpha} = \sum_{n=0}^{\infty}\left(\chi_n^{(\alpha)}\right)^{2m}\left|\hat{u}_n^{(\alpha)}\right|^2, \qquad \left\|\mathcal{D}_x^{m+1/2}u\right\|^2_{\omega_\alpha} = \sum_{n=0}^{\infty}\left(\chi_n^{(\alpha)}\right)^{2m+1}\left|\hat{u}_n^{(\alpha)}\right|^2. \tag{4.11}$$

Therefore, the norm of the space $\tilde{H}^r_{\omega_\alpha,c}(I)$ with real $r \geqslant 0$ can be characterized as

$$\|u\|_{\tilde{H}^r_{\omega_\alpha,c}} = \left(\sum_{n=0}^{\infty}\left(\chi_n^{(\alpha)}\right)^r\left|\hat{u}_n^{(\alpha)}\right|^2\right)^{\frac{1}{2}}, \quad \alpha > -1,\ c > 0. \tag{4.12}$$

An upper bound of the norm expressed in terms of the derivatives of u with explicit dependence on c, will be given in the end of this section.

The fundamental approximation result is stated below.

Theorem 4.1. *For any $u \in \tilde{H}^r_{\omega_\alpha,c}(I)$ with $r \geqslant 0$,*

$$\left\|\pi_{N,c}^{(\alpha)}u - u\right\|_{\omega_\alpha} \leqslant \left(\chi_{N+1}^{(\alpha)}\right)^{-\frac{r}{2}}\|u\|_{\tilde{H}^r_{\omega_\alpha,c}} \leqslant N^{-r}\|u\|_{\tilde{H}^r_{\omega_\alpha,c}}, \tag{4.13}$$

and in general, for $0 \leqslant \mu \leqslant r$,

$$\left\|\pi_{N,c}^{(\alpha)}u - u\right\|_{\tilde{H}^\mu_{\omega_\alpha,c}} \leqslant \left(\chi_{N+1}^{(\alpha)}\right)^{\frac{\mu-r}{2}}\left\|\pi_{N,c}^{(\alpha)}u - u\right\|_{\tilde{H}^r_{\omega_\alpha,c}} \leqslant \left(\chi_{N+1}^{(\alpha)}\right)^{\frac{\mu-r}{2}}\|u\|_{\tilde{H}^r_{\omega_\alpha,c}} \leqslant N^{\mu-r}\|u\|_{\tilde{H}^r_{\omega_\alpha,c}}. \tag{4.14}$$

Proof. We first assume that $r = 2m$. Since $\mathcal{D}_x\psi_n^{(\alpha)} = \chi_n^{(\alpha)}\psi_n^{(\alpha)}$, we derive from (3.2) that

$$\begin{aligned}\hat{u}_n^{(\alpha)} &= \int_{-1}^{1} u(x)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx = \frac{1}{(\chi_n^{(\alpha)})^m}\int_{-1}^{1} u(x)\mathcal{D}_x^m\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx \\ &= \frac{1}{(\chi_n^{(\alpha)})^m}\int_{-1}^{1}\mathcal{D}_x^m u(x)\psi_n^{(\alpha)}(x;c)\omega_\alpha(x)\,dx = \frac{1}{(\chi_n^{(\alpha)})^m}\left(\widehat{\mathcal{D}_x^m u}\right)_n,\end{aligned} \tag{4.15}$$

where $(\widehat{\mathcal{D}_x^m u})_n$ is the $(n+1)$th coefficient of the GPSWF expansion of $\mathcal{D}_x^m u$. Therefore,

$$\begin{aligned}\left\|\pi_{N,c}^{(\alpha)}u - u\right\|^2_{\omega_\alpha} &= \sum_{n=N+1}^{\infty}\left|\hat{u}_n^{(\alpha)}\right|^2 \overset{(4.15)}{=} \sum_{n=N+1}^{\infty}\left(\chi_n^{(\alpha)}\right)^{-2m}\left|\left(\widehat{\mathcal{D}_x^m u}\right)_n\right|^2 \\ &\overset{(3.6)}{\leqslant} \max_{n>N}\left\{\left(\chi_n^{(\alpha)}\right)^{-2m}\right\}\left\|\mathcal{D}_x^m u\right\|^2_{\omega_\alpha} \overset{(3.5)}{\leqslant} \left(\chi_{N+1}^{(\alpha)}\right)^{-2m}\|u\|^2_{\tilde{H}^{2m}_{\omega_\alpha,c}},\end{aligned}$$

which, together with (3.8), yields (4.13) with $r = 2m$.

We now prove (4.13) with $r = 2m+1$. By (3.6) and (4.12),

$$\begin{aligned}\left\|\pi_{N,c}^{(\alpha)}u - u\right\|^2_{\omega_\alpha} &= \sum_{n=N+1}^{\infty}\left|\hat{u}_n^{(\alpha)}\right|^2 \leqslant \max_{n>N}\left\{\left(\chi_n^{(\alpha)}\right)^{-(2m+1)}\right\}\sum_{n=N+1}^{\infty}\left(\chi_n^{(\alpha)}\right)^{2m+1}\left|\hat{u}_n^{(\alpha)}\right|^2 \\ &\leqslant \left(\chi_{N+1}^{(\alpha)}\right)^{-(2m+1)}\mathcal{A}_c^{(\alpha)}\left(\mathcal{D}_x^m u, \mathcal{D}_x^m u\right) = \left(\chi_{N+1}^{(\alpha)}\right)^{-(2m+1)}\|u\|^2_{\tilde{H}^{2m+1}_{\omega_\alpha,c}}.\end{aligned} \tag{4.16}$$

The above two estimates, together with a space interpolation, lead to (4.13).

To prove (4.14), we obtain from (3.6), (3.8) and (4.12) that

$$\left\|\pi_{N,c}^{(\alpha)}u - u\right\|^2_{\tilde{H}^\mu_{\omega_\alpha,c}} = \sum_{n=N+1}^{\infty}\left(\chi_n^{(\alpha)}\right)^{\mu}|\hat{u}_n|^2 \leqslant \max_{n>N}\left\{\left(\chi_n^{(\alpha)}\right)^{\mu-r}\right\}\sum_{n=N+1}^{\infty}\left(\chi_n^{(\alpha)}\right)^{r}|\hat{u}_n|^2$$

$$= \big(\chi_{N+1}^{(\alpha)}\big)^{\mu-r} \big\|\pi_{N,c}^{(\alpha)} u - u\big\|^2_{\widetilde{H}^r_{\omega_\alpha,c}} \leqslant \big(\chi_{N+1}^{(\alpha)}\big)^{\mu-r} \sum_{n=0}^{\infty} \big(\chi_n^{(\alpha)}\big)^r |\hat{u}_n|^2$$
$$= \big(\chi_{N+1}^{(\alpha)}\big)^{\mu-r} \|u\|^2_{\widetilde{H}^r_{\omega_\alpha,c}} \leqslant N^{2(\mu-r)} \|u\|^2_{\widetilde{H}^r_{\omega_\alpha,c}}.$$

This completes the proof. □

The norm in the upper bounds of the above estimates is expressed in the terms of the expansion coefficients, and implicitly depends on the tuning parameter c. It is preferable to represent it in terms of the derivatives of u, and more importantly, to explore the explicit dependence of the convergence on the parameter c. Notice that letting $c = 0$ in Theorem 4.1 (i.e., the Gegenbauer approximations [27]), we have

$$\|u\|^2_{\widetilde{H}^r_{\omega_\alpha,0}} \leqslant C \sum_{k=0}^{r} \int_{-1}^{1} \big|\partial_x^k u\big|^2 \big(1-x^2\big)^{\alpha+k}\, dx, \tag{4.17}$$

where C is a positive constant independent of u. Using integration by parts, we obtain that for $r = 1, 2, 3$,

$$\|u\|^2_{\widetilde{H}^r_{\omega_\alpha,c}} \leqslant C \sum_{k=0}^{r} \bar{c}^{2(r-k)} \int_{-1}^{1} \big|\partial_x^k u\big|^2 \big(1-x^2\big)^{\alpha+k}\, dx, \quad \bar{c} = \max\{1, c\}, \tag{4.18}$$

where C is the uniform upper bound of some polynomials of x, α and $\bar{c}^{-1}$. However, the presence of the term c^2x^2 makes it very complicated and tedious to verify such a sharp bound (in terms of the weight $(1-x^2)^{\alpha+k}$) for all $r \geqslant 4$. Nevertheless, a considerably rough bound can be derived from a direct calculation and an induction. More precisely, we have that

$$\mathcal{D}_x^m u = \sum_{k=0}^{2m} \bar{c}^{2[k/2]} p_k\big(x; \bar{c}^{-2}, \alpha\big) \big(1-x^2\big)^{(m-k)_+} \partial_x^{2m-k} u, \tag{4.19}$$

where $(m-k)_+ = \max\{m-k, 0\}$, $[a]$ denotes the maximum integer $\leqslant a$, and $\{p_k(x; \bar{c}^{-2}, \alpha)\}$ are some generic polynomials of degree $\leqslant m$ with coefficients involving $\bar{c}^{-2}$ and α. Therefore,

$$\|u\|_{\widetilde{H}^{2m}_{\omega_\alpha,c}} = \big\|\mathcal{D}_x^m u\big\|_{\omega_\alpha} \leqslant C \sum_{k=0}^{2m} c^{[k/2]} \big\| \big(1-x^2\big)^{(m-k)_+} \partial_x^{2m-k} u \big\|_{\omega_\alpha}, \tag{4.20}$$

where C is a positive constant independent of c and u. A similar upper bound can be obtained for $r = 2m+1$. Moreover, there holds $H^r_{\omega_\alpha}(I) \subseteq \widetilde{H}^r_{\omega_\alpha,c}(I)$, and

$$\|u\|_{\widetilde{H}^r_{\omega_\alpha,c}} \leqslant C\big(1+c^2\big)^{r/2} \|u\|_{H^r_{\omega_\alpha}}. \tag{4.21}$$

4.2. GPSWF spectral methods

As an important application, we construct collocation and spectral-Galerkin methods using GPSWFs as basis functions and demonstrate the advantages in the enhancement of spatial accuracy and resolution. As with the Slepian functions, they can significantly relax time-constraint of the explicit time-stepping to be reported in the forthcoming paper [59].

4.2.1. Collocation scheme

As with a polynomial-based collocation method, it is necessary to associate the method with an appropriate quadrature formula. Following the rule for the case: $\alpha = 0$ (cf. [12]), we choose the quadrature points and weights $\{\zeta_{N,j}^{(\alpha)}, \sigma_{N,j}^{(\alpha)}\}_{j=0}^{N}$ (with $\zeta_{N,0}^{(\alpha)} = -1$ and $\zeta_{N,N}^{(\alpha)} = 1$), such that

$$\int_{-1}^{1} \psi_n^{(\alpha)}(x; c)\omega_\alpha(x)\, dx = \sum_{j=0}^{N} \psi_n^{(\alpha)}\big(\zeta_{N,j}^{(\alpha)}; c\big)\sigma_{N,j}^{(\alpha)}, \quad n = 0, 1, \ldots, 2N-1. \tag{4.22}$$

The points and weights can be numerically evaluated by using Newton's iteration as in [12]. It's interesting to point out that Beylkin and Monzón [7] considered new generalized Gaussian quadratures involving exponentials and weight functions supported in $\bar{I}$, for integration and interpolation of bandlimited functions.

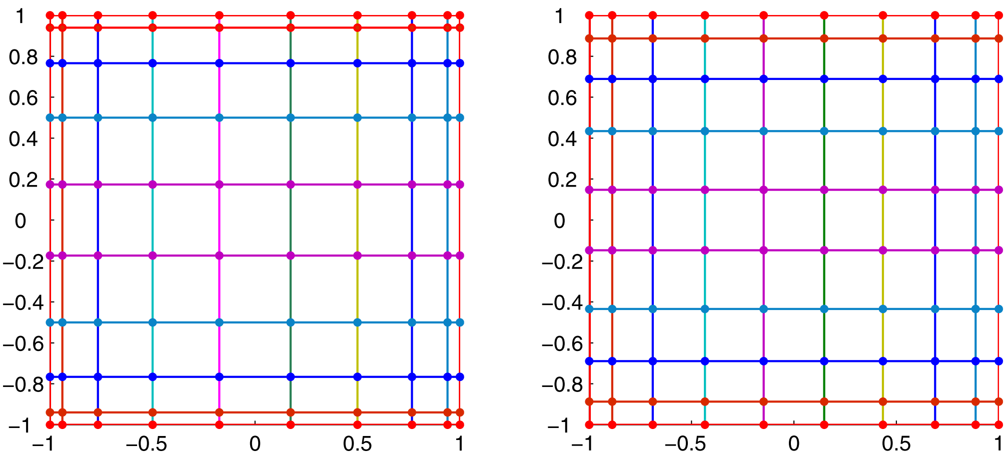

Fig. 4.1. Distribution of the tensorial Lobatto points: $N=9$ and $\alpha=-0.5$. Left: $c=0$, and right: $c=8$.

Let $X_{N,c}^{(\alpha)}$ be the finite-dimensional space as in (4.1), and define the Lagrange–GPSWF nodal basis as

$$h_j^{(\alpha)}(x;c)=\sum_{k=0}^{N} d_k^j \psi_k^{(\alpha)}(x;c)\in X_{N,c}^{(\alpha)},\quad j=0,1,\dots,N, \tag{4.23}$$

where the coefficients $\{d_k^j\}$ are uniquely determined by $h_j^{(\alpha)}(\zeta_{N,k}^{(\alpha)};c)=\delta_{jk}$, $0\leqslant j,k\leqslant N$.

Equipped with the nodal basis (4.23), the GPSWF collocation methods can be implemented essentially in the same fashion as the Legendre and Chebyshev methods (see, e.g., [13,28]). More precisely, for any $u\in X_{N,c}^{(\alpha)}$, we write

$$u(x)=\sum_{j=0}^{N} u\big(\zeta_{N,j}^{(\alpha)}\big)h_j^{(\alpha)}(x;c), \tag{4.24}$$

and define the differentiation matrices

$$\boldsymbol{D}^{(k)}=\big(\partial_x^k h_j^{(\alpha)}\big(\zeta_{N,i}^{(\alpha)};c\big)\big)_{0\leqslant i,j\leqslant N},\quad k=1,2,\dots. \tag{4.25}$$

Hence, the numerical derivatives can be computed by the matrix–vector multiplication as usual.

In Fig. 4.1, we plot two-dimensional 10×10 tensorial Chebyshev–Lobatto points and the GPSWF–Lobatto points with $\alpha=-0.5$ and $c=8$. We see that the latter case is more uniformly distributed. This is a key advantage of the GPSWF collocation method over the polynomial-based method. Among other things, the new approach requires fewer grid points to achieve a prescribed accuracy. Furthermore, the differentiation matrix has a condition number of order $O(N^{3/2})$ (vs. $O(N^2)$ for the usual polynomial interpolation), which significantly relaxes the restriction of explicit time-stepping method for large values of N (cf. [14,11,59]).

4.2.2. *Spectral-Galerkin scheme*

We next propose and analyze a spectral-Galerkin method using a modal basis consisting of a compact linear combination of integration of the GPSWFs, which leads to better-conditioned systems. It is worthwhile to point out that the collocation/pseudospectral methods using Slepian basis have been employed by a number of authors, but the Galerkin approach has only little explored.

To fix the main idea, we start with the one-dimensional model equation:

$$-u''(x)+\gamma u(x)=f(x)\quad \text{in } I=(-1,1),\qquad u(\pm 1)=0, \tag{4.26}$$

where $\gamma\geqslant 0$ and f is a given function. Let $H_{0,\omega_\alpha}^1(I)=\{u\in H_{\omega_\alpha}^1(I):\ u(\pm 1)=0\}$. The variational formulation of (4.26) is to find $u\in H_{0,\omega_\alpha}^1(I)$, such that

$$a_\alpha(u,v)=\big(\partial_x u,\partial_x(v\omega_\alpha)\big)+\gamma(u,v)_{\omega_\alpha}=(f,v)_{\omega_\alpha},\quad \forall v\in H_{0,\omega_\alpha}^1(I). \tag{4.27}$$

For $|\alpha|<1$, the bilinear form $a_\alpha(\cdot,\cdot)$ is continuous and coercive in $H_{0,\omega_\alpha}^1(I)$ (cf. [4]). Hence, if $f\in L_{\omega_\alpha}^2(I)$, the problem (4.27) admits a unique solution in $H_{0,\omega_\alpha}^1(I)$.

Define

$$\Phi_j^{(\alpha)}(x;c) = a_j \Psi_j^{(\alpha)}(x;c) + b_j \Psi_{j+2}^{(\alpha)}(x;c), \quad j \geqslant 0, \tag{4.28}$$

where

$$\Psi_j^{(\alpha)}(x;c) = \int_{-1}^{x} \psi_j^{(\alpha)}(t;c)\,dt, \quad j \geqslant 0,$$

and the coefficients $\{a_j, b_j\}$ are chosen such that $a_j = 1$ and $b_j = 0$ for odd j, and

$$a_j = \frac{\Psi_{j+2}^{(\alpha)}(1;c)}{\sqrt{[\Psi_j^{(\alpha)}(1;c)]^2 + [\Psi_{j+2}^{(\alpha)}(1;c)]^2}}, \qquad b_j = -\frac{\Psi_j^{(\alpha)}(1;c)}{\sqrt{[\Psi_j^{(\alpha)}(1;c)]^2 + [\Psi_{j+2}^{(\alpha)}(1;c)]^2}}, \quad \text{for even } j. \tag{4.29}$$

We notice that $\Phi_j^{(\alpha)}(\pm 1) = 0$ for all $j \geqslant 0$.

Remark 4.1. For $\alpha = 0, -\frac{1}{2}$, compact linear combinations of Legendre and Chebyshev polynomials have been used for spectral-Galerkin methods (cf. [39,40]), and the underlying linear systems are well conditioned and particularly sparse for problems with constant or polynomial coefficients. The basis (4.28)–(4.29) can be viewed as an extension of such type of basis functions.

Define the finite-dimensional approximation space

$$Y_{N,c}^{(\alpha)} = \operatorname{span}\{\Phi_0^{(\alpha)}, \Phi_1^{(\alpha)}, \ldots, \Phi_{N-2}^{(\alpha)}\}. \tag{4.30}$$

The GPSWF spectral-Galerkin approximation to (4.26) is to find $u_N \in Y_{N,c}^{(\alpha)}$ such that

$$a_\alpha(u_N, v_N) = (f, v_N)_{\omega_\alpha}, \quad \forall v_N \in Y_{N,c}^{(\alpha)}. \tag{4.31}$$

Next, we perform an error analysis of the proposed scheme. For any $u \in H^1_{0,\omega_\alpha}(I)$, we write

$$u(x) = \sum_{n=0}^{\infty} \tilde{u}_n \Phi_n^{(\alpha)}(x;c) = \sum_{n=0}^{\infty} (a_n \tilde{u}_n + b_{n-2}\tilde{u}_{n-2}) \Psi_n^{(\alpha)}(x;c), \tag{4.32}$$

where $\tilde{u}_{-2} = \tilde{u}_{-1} = 0$. For $r \geqslant 1$, we define

$$\widehat{H}^r_{\omega_\alpha,c}(I) := \left\{ u \in H^1_{0,\omega_\alpha}(I) \colon \|u\|_{\widehat{H}^r_{\omega_\alpha,c}} = \left(\sum_{n=0}^{\infty} \big(\chi_n^{(\alpha)}\big)^{r-1} |\tilde{u}_n|^2 \right)^{1/2} < \infty \right\}. \tag{4.33}$$

Since $|a_n|, |b_n| \leqslant 1$, one verifies from (4.12) that for any $u \in \widehat{H}^r_{\omega_\alpha,c}(I)$ with $r \geqslant 1$, we have $\partial_x u \in \widetilde{H}^{r-1}_{\omega_\alpha,c}(I)$.
The convergence result is stated below.

Theorem 4.2. *Let u and u_N be the solutions of* (4.26) *and* (4.31)*, respectively. If $u \in \widehat{H}^r_{\omega_\alpha,c}(I)$ with $r \geqslant 1$, then*

$$\|u - u_N\|_{1,\omega_\alpha} \leqslant C N^{1-r} \|u\|_{\widehat{H}^r_{\omega_\alpha,c}}, \tag{4.34}$$

where C is a generic constant independent of c, N and u.

Proof. Using a standard argument for Galerkin methods leads to

$$\|u - u_N\|_{1,\omega_\alpha} \leqslant C \inf_{v_N \in Y_{N,c}^{(\alpha)}} \|u - v_N\|_{1,\omega_\alpha}. \tag{4.35}$$

Setting $v_N = \sum_{n=0}^{N-2} \tilde{u}_n \Phi_n^{(\alpha)}$, we find

$$\partial_x(u - v_N) = \sum_{n=N-1}^{\infty} \tilde{u}_n \Phi_n^{(\alpha)} = a_{N-1}\tilde{u}_{N-1}\psi_{N-1}^{(\alpha)} + a_N \tilde{u}_N \psi_N^{(\alpha)} + \sum_{n=N+1}^{\infty} (a_n \tilde{u}_n + b_{n-2}\tilde{u}_{n-2}) \psi_n^{(\alpha)}.$$

Hence, by the orthogonality (3.5) and the fact $|a_n|, |b_n| \leqslant 1$,

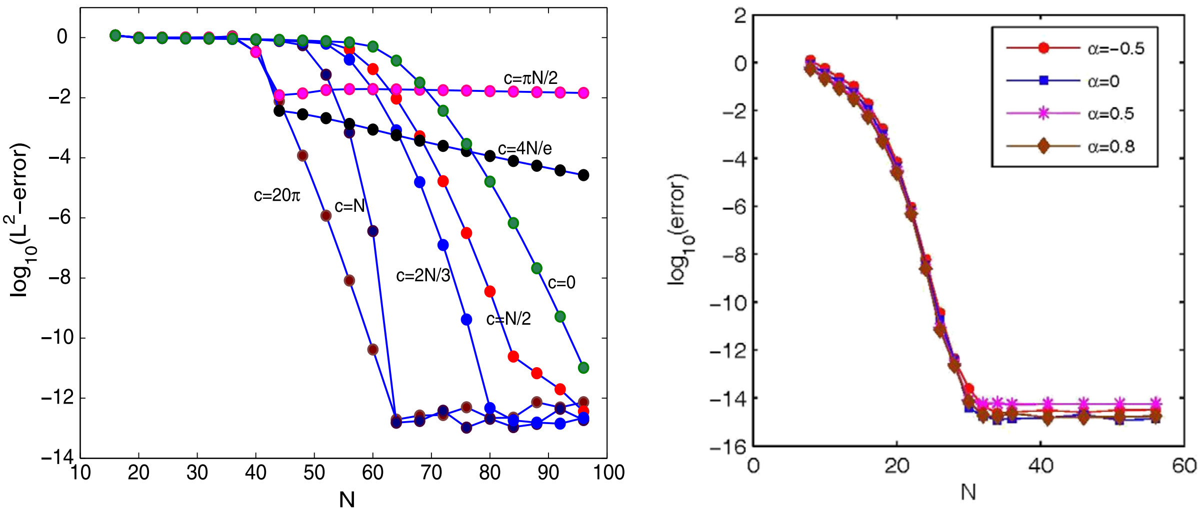

Fig. 4.2. Example 1 (left): $\log_{10}(L^2\text{-error})$ against N for various $c = 0,\ 20\pi,\ N/2,\ 2N/3,\ N,\ 4N/e,\ \pi N/2$. Example 2 (right): $\log_{10}(L^2\text{-error})$ against N with $\alpha = -0.5,\ 0,\ 0.5,\ 0.8$ and $c = N/2$.

$$\big\|\partial_x(u - v_N)\big\|^2_{\omega_\alpha} = |a_{N-1}\tilde{u}_{N-1}|^2 + |a_N\tilde{u}_N|^2 + \sum_{n=N+1}^{\infty} |a_n\tilde{u}_n + b_{n-2}\tilde{u}_{n-2}|^2$$

$$\leqslant 2\sum_{n=N-1}^{\infty} |\tilde{u}_n|^2 \overset{(3.6)}{\leqslant} 2\big(\chi_{N-1}^{(\alpha)}\big)^{1-r} \sum_{n=N-1}^{\infty} \big(\chi_n^{(\alpha)}\big)^{r-1} |\tilde{u}_n|^2 \overset{(4.33)}{\leqslant} CN^{2(1-r)} \|u\|^2_{\widehat{H}^r_{\omega_\alpha,c}}.$$

Using the Poincaré inequality (cf. [4]): $\|u\|_{\omega_\alpha} \leqslant C\|\partial_x u\|_{\omega_\alpha}$, the desired result follows from (4.35) and the above estimate. □

4.3. Numerical results

Now, we present some numerical results to support our theoretical results and demonstrate the advantages of the GPSWF methods.

We first consider (4.26) with $\gamma = 1$ and the exact solution: $u(x) = \sin(20\pi x)$. We use the Galerkin scheme (4.27) with $\alpha = 0$ for the following computation. In Fig. 4.2 we plot $\log_{10}(L^2\text{-error})$ against $N \in [8, 96]$ for various $c = 0, 20\pi, N/2, 2N/3, N,\ 4N/e, \pi N/2$. It indicates that the GPSWF approximation with $c = 20\pi$ provides the best result as predicted in (4.6), and a more rapid convergence is observed for $c = \delta N$ as δ increases from 0 to 1. However, the accuracy deteriorates for $c > N$. We also see that the GPSWFs with a suitable bandwidth c produce a better result than Legendre approximations ($c = 0$) particularly for high-oscillatory smooth solutions. This agrees well with the analysis in Theorems 4.1 and 4.2.

We next consider the two-dimensional Poisson equation

$$-\Delta u = f \quad \text{in } \Omega = (-1, 1)^2, \qquad u|_{\partial\Omega} = 0, \tag{4.36}$$

and the exact solution: $u(x, y) = \sin(4\pi x)\sin(3\pi y)\exp(xy)$. In the first test, we solve this problem by the collocation and Galerkin methods using a tensor product of the basis functions in (4.28)–(4.29) with $\alpha = -0.5$, respectively. In Fig. 4.3, we depict $\log_{10}$ of the maximum point-wise errors (marked by square) and L^2-errors (marked by circle) against $N \in [8, 56]$ for various $c = 0, N/2, N$ by using GPSWF–collocation method (left) and GPSWF–Galerkin method (right). We see that both approaches provide spectrally accurate approximations, and more rapid convergence is observed for $c = N/2$. In practice, we choose $c = N/2$ to obtain quasi-uniform grids and achieve high order of accuracy. On the other hand, we see that the collocation method suffers from a severer round-off error due to the conditioning of the system.

In Fig. 4.2 (right), we plot $\log_{10}$ of L^2-errors (marked by circle) against $N \in [8, 56]$ for $c = N/2$ and $\alpha = -0.5, 0, 0.5, 0.8$, and a similar convergence behavior is observed for various choices of $-1 < \alpha < 1$.

5. Concluding discussions

In this paper, we defined a family of GPSWFs, which extends the Slepian functions to real order $\alpha > -1$, and also generalizes the Gegenbauer polynomials an orthogonal system equipped with an intrinsic tuning parameter $c > 0$. Notably, being defined as the eigenfunctions of a Sturm–Liouville operator, we showed that the GPSWFs also satisfy an integral equation. This led to some important analytic and asymptotic formulas for the GPSWFs and the associated eigenvalues.

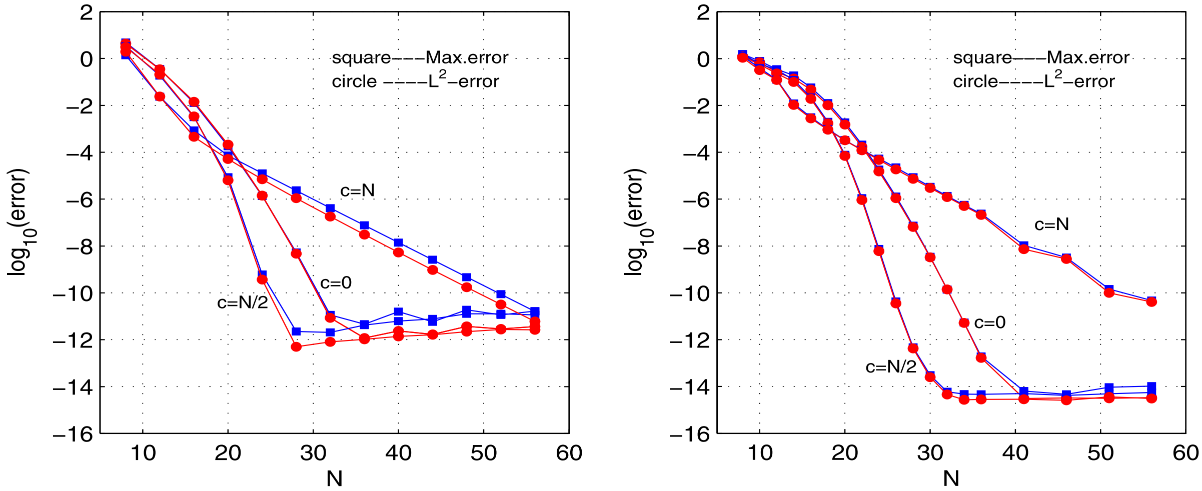

Fig. 4.3. Example 2: $\log_{10}$ of the maximum point-wise errors (marked by square) and L^2-errors (marked by circle) against N for $c = 0$, $N/2$, N with $\alpha = -0.5$. Left: collocation method and right: spectral-Galerkin method.

We analyzed the approximation properties of the GPSWF expansions for functions in weighted Sobolev spaces. The approximation results are optimal and featured with explicit dependence on the parameter c. We implemented the GPSWF collocation and Galerkin methods for model elliptic equations in one and two dimensions, and provided some guidelines on the choice of c to achieve better approximations than the polynomial-based methods. The use of such basis functions to develop efficient high-order spectral-element methods on quasi-uniform grids for solving shallow water equations on spherical geometry will be reported in the forthcoming paper [59].

We are grateful to the referees for bringing our attentions to the book [29], where one example on page 201 may motivate us to consider the Sturm–Liouville problem:

$$-(1-x^2)u'' + \{\beta - \alpha + (\alpha + \beta + 2)x\}u' + \{p(\beta - \alpha)x - p^2x^2\}u = \chi u, \quad \alpha, \beta > -1, \ x \in (-1, 1),$$

where p is a constant. Its eigenfunction satisfies the integral equation

$$\lambda u(x) = \int_{-1}^{1} e^{pxt}(1-t)^{\alpha}(1+t)^{\beta}u(t)\,dt.$$

We realize that when $\alpha = \beta$ and $p = \mathrm{i}c$, it reduces to (3.10). Hence, it is possible to extend the arguments in this paper to this more general case.

Acknowledgments

The authors would like to thank the editor and referees for their thoughtful comments, and their suggestions for adding several important references.

Appendix A. Proof of Lemma 2.1

We carry out the proof by induction. The formula with $n = 0$ is trivial.

For $n = 1$, we obtain from (2.3) and (2.10) that

$$\begin{aligned} G_1^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] &\overset{(2.3)}{=} \frac{\sqrt{(2\alpha+3)\Gamma(2\alpha+2)}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} \cdot \frac{1}{\mathrm{i}}\frac{d}{dz}\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] \\ &= \frac{\sqrt{(2\alpha+3)\Gamma(2\alpha+2)}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\mathrm{i}} \cdot \frac{\partial_z J_{\alpha+\frac{1}{2}}(z) - (\alpha+\frac{1}{2})z^{\alpha-\frac{1}{2}}J_{\alpha+\frac{1}{2}}(z)}{z^{2\alpha+1}} \\ &\overset{(2.10)}{=} \frac{\sqrt{(2\alpha+3)\Gamma(2\alpha+2)}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\mathrm{i}} \end{aligned}$$

$$\cdot \frac{(\alpha+\frac{1}{2})z^{\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}(z) - z^{\alpha+\frac{1}{2}} J_{\alpha+\frac{3}{2}}(z) - (\alpha+\frac{1}{2})z^{\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}(z)}{z^{2\alpha+1}}$$
$$= \mathrm{i}\frac{\sqrt{(2\alpha+3)\Gamma(2\alpha+2)}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} \frac{J_{\alpha+\frac{3}{2}}(z)}{z^{\alpha+\frac{1}{2}}} = \mathrm{i}h_1^{(\alpha)} \frac{J_{\alpha+\frac{3}{2}}(z)}{z^{\alpha+\frac{1}{2}}}.$$

This implies the desired result with $n = 1$.

Assuming the formula (2.14) is true for $n = N-1, N$, it suffices to prove this formula also holds for $n = N+1$. By the three-term recurrence relation (2.3),

$$G_{N+1}^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right) = -\mathrm{i}a_N \frac{d}{dz}\left[G_N^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\right] - b_N\, G_{N-1}^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right),$$

where a_N and b_N are defined in (2.4). Using the inductive hypothesis gives

$$\begin{aligned}
&G_{N+1}^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] \\
&= -\mathrm{i}a_N \frac{d}{dz} G_N^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] - b_N G_{N-1}^{(\alpha)}\left(\frac{1}{\mathrm{i}}\frac{d}{dz}\right)\left[\frac{J_{\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] \\
&= -\mathrm{i}^{N+1} a_N h_N^{(\alpha)} \frac{d}{dz}\left[\frac{J_{N+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right] - \mathrm{i}^{N-1} b_N h_{N-1}^{(\alpha)} \frac{J_{N+\alpha-\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} \\
&= \mathrm{i}^{N+1}\left\{b_N h_{N-1}^{(\alpha)} \frac{J_{N+\alpha-\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} - a_N h_N^{(\alpha)}\left(\frac{J'_{N+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} - \left(\alpha+\frac{1}{2}\right)\frac{J_{N+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{3}{2}}}\right)\right\} \\
&\overset{(2.10)}{=} \mathrm{i}^{N+1}\left\{b_N h_{N-1}^{(\alpha)} \frac{J_{N+\alpha-\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} - a_N h_N^{(\alpha)}\left(\frac{J_{N+\alpha-\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} - (N+2\alpha+1)\frac{J_{N+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{3}{2}}}\right)\right\} \\
&\overset{(2.11)}{=} \mathrm{i}^{N+1}\left\{a_N h_N^{(\alpha)} \frac{N+2\alpha+1}{2N+2\alpha+1} \frac{J_{N+\alpha+\frac{3}{2}}(z)}{z^{\alpha+\frac{1}{2}}} + \left(b_N h_{N-1}^{(\alpha)} - a_N h_N^{(\alpha)} \frac{N}{2N+2\alpha+1}\right)\frac{J_{N+\alpha-\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}}\right\}.
\end{aligned}$$

A direct calculation by using (2.4) yields

$$h_{N+1}^{(\alpha)} = a_N h_N^{(\alpha)} \frac{N+2\alpha+1}{2N+2\alpha+1}, \qquad b_N h_{N-1}^{(\alpha)} - a_N h_N^{(\alpha)} \frac{N}{2N+2\alpha+1} = 0.$$

Hence the desired formula holds for $n = N+1$, which completes the induction.

Appendix B. Proof of Lemma 3.1

Differentiating Eq. (3.3) with respect to c yields

$$\partial_x\big(\omega_{\alpha+1}\partial_x\partial_c\psi_n^{(\alpha)}\big) + \big(\chi_n^{(\alpha)}(c) - c^2x^2\big)\omega_\alpha\partial_c\psi_n^{(\alpha)} = \big(2cx^2 - \partial_c\chi_n^{(\alpha)}(c)\big)\omega_\alpha\psi_n^{(\alpha)}.$$

Multiplying the above equation by $\psi_n^{(\alpha)}$, and integrating the resulting equation over $(-1,1)$, we derive from (3.3) and integrating by parts that

$$\begin{aligned}
&2c\int_{-1}^{1} x^2\big[\psi_n^{(\alpha)}(x;c)\big]^2\omega_\alpha(x)\,dx - \frac{\partial\chi_n^{(\alpha)}}{\partial c} \\
&= \int_{-1}^{1}\big\{\partial_x\big(\omega_{\alpha+1}\partial_x\partial_c\psi_n^{(\alpha)}(x;c)\big) + \big(\chi_n^{(\alpha)}(c) - c^2x^2\big)\omega_\alpha\partial_c\psi_n^{(\alpha)}(x;c)\big\}\psi_n^{(\alpha)}(x;c)\,dx \\
&= \int_{-1}^{1}\big\{\partial_x\big(\omega_{\alpha+1}\partial_x\psi_n^{(\alpha)}(x;c)\big) + \big(\chi_n^{(\alpha)}(c) - c^2x^2\big)\omega_\alpha\psi_n^{(\alpha)}(x;c)\big\}\partial_c\psi_n^{(\alpha)}(x;c)\,dx = 0,
\end{aligned}$$

which, together with (3.5), implies that

$$0<\frac{\partial \chi_n^{(\alpha)}}{\partial c}=2c\int_{-1}^{1} x^2\big[\psi_n^{(\alpha)}(x;c)\big]^2\omega_\alpha(x)\,dx<2c \quad\Rightarrow\quad 0<\chi_n^{(\alpha)}(c)-\chi_n^{(\alpha)}(0)<c^2.$$

Since $\chi_n^{(\alpha)}(0)=n(n+2\alpha+1)$, the desired result follows.

Appendix C. Proof of Lemma 3.2

Following the general perturbation scheme in [44], we expand the eigen-pair $\{\chi_n^{(\alpha)}(c),\psi_n^{(\alpha)}(x;c)\}$ in series of c^2:

$$\psi_n^{(\alpha)}(x;c)=G_n^{(\alpha)}(x)+\sum_{j=1}^{\infty}c^{2j}Q_{n,j}(x,\alpha),\qquad \chi_n^{(\alpha)}(c)=\gamma_n^{(\alpha)}+\sum_{j=1}^{\infty}c^{2j}a_{n,j}(\alpha), \tag{C.1}$$

where $\gamma_n^{(\alpha)}=\chi_n^{(\alpha)}(0)$ (cf. (3.4)), and

$$Q_{n,j}(x,\alpha)=\sum_{k=-j}^{j}B_{2k,n}(j,\alpha)G_{n+2k}^{(\alpha)}(x), \tag{C.2}$$

with $B_{0,n}=0$. Let $\mathcal{L}_x^{(\alpha)}$ and $\mathcal{D}_x$ be the Sturm–Liouville operators defined in (2.1) and (3.1), respectively. Substituting the expansion (C.1) into

$$\mathcal{D}_x\psi_n^{(\alpha)}(x;c)=\chi_n^{(\alpha)}(c)\psi_n^{(\alpha)}(x;c),\quad n\geqslant 1, \tag{C.3}$$

equating to zero the coefficients of distinct powers of c^2, we find the equation corresponding to the coefficient of c^2 is

$$\big(\mathcal{L}_x^{(\alpha)}-\gamma_n^{(\alpha)}\big)Q_{n,1}+x^2G_n^{(\alpha)}-a_{n,1}G_n^{(\alpha)}=0. \tag{C.4}$$

Hence, using $\mathcal{L}_x^{(\alpha)}G_n^{(\alpha)}=\gamma_n^{(\alpha)}G_n^{(\alpha)}$, and the property (2.3) and (C.2), we find

$$\begin{aligned} B_{2,n}&=\frac{e_n}{\gamma_{n+2}^{(\alpha)}-\gamma_n^{(\alpha)}}=\frac{1}{2(2n+2\alpha+3)^2}\sqrt{\frac{(n+1)(n+2)(n+2\alpha+1)(n+2\alpha+2)}{(2n+2\alpha+1)(2n+2\alpha+5)}},\\ a_{n,1}&=d_n=\frac{2n(n+2\alpha+1)+2\alpha-1}{(2n+2\alpha-1)(2n+2\alpha+3)}, \end{aligned} \tag{C.5}$$

and $B_{-2,n}=-B_{2,n-2}$, where e_n and d_n are defined in (3.40). In view of $B_{0,n}=0$, we obtain

$$\psi_n^{(\alpha)}(x;c)=G_n^{(\alpha)}(x)+c^2\big(B_{-2,n}G_{n-1}^{(\alpha)}(x)+B_{2,n}G_{n+2}^{(\alpha)}(x)\big)+O\big(c^4\big), \tag{C.6}$$

and

$$\chi_n^{(\alpha)}(c)=\gamma_n^{(\alpha)}+c^2a_{n,1}+O\big(c^4\big). \tag{C.7}$$

This ends the proof.

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1964.
[2] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[3] M.A. Al-Gwaiz, Sturm–Liouville Theory and Its Applications, Springer, 2007.
[4] C. Bernardi, Y. Maday, Spectral method, in: P.G. Ciarlet, L.L. Lions (Eds.), Handbook of Numerical Analysis, vol. 5, North-Holland, 1997 (Part 2).
[5] G. Beylkin, C. Kurcz, L. Monzón, Grids transforms for band-limited functions in a disk, Inverse Problems 23 (5) (2007) 2059–2088.
[6] G. Beylkin, L. Monzón, On generalized Gaussian quadratures for exponentials and their applications, Appl. Comput. Harmon. Anal. 12 (3) (2002) 332–373.
[7] G. Beylkin, K. Sandberg, Wave propagation using bases for bandlimited functions, Wave Motion 41 (3) (2005) 263–291.
[8] C.J. Bouwkamp, On the theory of spheroidal wave functions of order zero, Nederl. Akad. Wetensch. Proc. 53 (1950) 931–944, Indag. Math. 12 (1950) 326–339.
[9] J.P. Boyd, Approximation of an analytic function on a finite real interval by a bandlimited function and conjectures on properties of prolate spheroidal functions, Appl. Comput. Harmon. Anal. 15 (2) (2003) 168–176.
[10] J.P. Boyd, Large mode number eigenvalues of the prolate spheroidal differential equation, Appl. Math. Comput. 145 (2–3) (2003) 881–886.
[11] J.P. Boyd, Prolate spheroidal wavefunctions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudospectral algorithms, J. Comput. Phys. 199 (2) (2004) 688–716.
[12] J.P. Boyd, Algorithm 840: computation of grid points, quadrature weights and derivatives for spectral element methods using prolate spheroidal wave functions—prolate elements, ACM Trans. Math. Software 31 (1) (2005) 149–165.
[13] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods: Fundamentals in Single Domains (Scientific Computation), Springer-Verlag, Berlin, 2006.

[14] Q.-Y. Chen, D. Gottlieb, J.S. Hesthaven, Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, SIAM J. Numer. Anal. 43 (5) (2005) 1912–1933 (electronic).

[15] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[16] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. 1, Interscience Publishers, 1953.

[17] D. Dąbrowska, Recovering signals from inner products involving prolate spheroidals in the presence of jitter, Math. Comp. 74 (249) (2005) 279–290 (electronic).

[18] C. Flammer, Spheroidal Wave Functions, Stanford University Press, Stanford, CA, 1957.

[19] A. Gelb, Z. Jackiewicz, Determining analyticity for parameter optimization of the Gegenbauer reconstruction method, SIAM J. Sci. Comput. 27 (3) (2005) 1014–1031 (electronic).

[20] A. Glaser, X. Liu, V. Rokhlin, A fast algorithm for the calculation of the roots of special functions, SIAM J. Sci. Comput. 29 (4) (2007) 1420–1438 (electronic).

[21] D. Gottlieb, C.-W. Shu, On the Gibbs phenomenon. IV. Recovering exponential accuracy in a subinterval from a Gegenbauer partial sum of a piecewise analytic function, Math. Comp. 64 (211) (1995) 1081–1095.

[22] F.A. Grünbaum, Band time limiting, recursion relations and some nonlinear evolution equations, in: Special Functions: Group Theoretical Aspects and Applications, in: Math. Appl., Reidel, Dordrecht, 1984, pp. 271–286.

[23] F.A. Grünbaum, Differential operators commuting with convolution integral operators, J. Math. Anal. Appl. 91 (1) (1983) 80–93.

[24] F.A. Grünbaum, L. Longhi, M. Perlstadt, Differential operators commuting with finite convolution integral operators: some nonabelian examples, SIAM J. Appl. Math. 42 (5) (1982) 941–955.

[25] F.A. Grünbaum, L. Miranian, Magic of the prolate spheroidal functions in various setups, in: Proceedings of SPIE, vol. 4478, 2001, p. 151.

[26] B.-Y. Guo, Gegenbauer approximation in certain Hilbert spaces and its applications to singular differential equations, SIAM J. Numer. Anal. 37 (2) (2000) 621–645 (electronic).

[27] B.-Y. Guo, L.-L. Wang, Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces, J. Approx. Theory 128 (1) (2004) 1–41.

[28] J. Hesthaven, S. Gottlieb, D. Gottlieb, Spectral Methods for Time-Dependent Problems, Cambridge Monographs on Applied and Computational Mathematics, Cambridge Univ. Press, Cambridge, 2007.

[29] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.

[30] A. Karoui, T. Moumni, New efficient methods of computing the prolate spheroidal wave functions and their corresponding eigenvalues, Appl. Comput. Harmon. Anal. 24 (2008) 269–289.

[31] N. Kovvali, W. Lin, L. Carin, Pseudospectral method based on prolate spheroidal wave functions for frequency-domain electromagnetic simulations, IEEE Trans. Antennas and Propagation 53 (2005) 3990–4000.

[32] N. Kovvali, W. Lin, Z. Zhao, L. Couchman, L. Carin, Rapid prolate pseudospectral differentiation and interpolation with the fast multipole method, SIAM J. Sci. Comput. 28 (2) (2006) 485–497 (electronic).

[33] H.J. Landau, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. II, Bell System Tech. J. 40 (1961) 65–84.

[34] H.J. Landau, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals, Bell System Tech. J. 41 (1962) 1295–1336.

[35] N.W. McLachlan, Theory and Applications of Mathieu Functions, Oxford Press, London, 1951.

[36] I.C. Moore, M. Cada, Prolate spheroidal wave functions, an introduction to the Slepian series and its properties, Appl. Comput. Harmon. Anal. 16 (3) (2004) 208–230.

[37] S.-C. Pei, J.-J. Ding, Generalized prolate spheroidal wave functions for optical finite fractional Fourier and linear canonical transforms, J. Opt. Soc. Amer. A 22 (3) (2005) 460–474.

[38] V. Rokhlin, H. Xiao, Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit, Appl. Comput. Harmon. Anal. 22 (1) (2007) 105–123.

[39] J. Shen, Efficient spectral-Galerkin method I. Direct solvers for second- and fourth-order equations by using Legendre polynomials, SIAM J. Sci. Comput. 15 (1994) 1489–1505.

[40] J. Shen, Efficient spectral-Galerkin method II. Direct solvers for second- and fourth-order equations by using Chebyshev polynomials, SIAM J. Sci. Comput. 16 (1995) 74–87.

[41] J. Shen, L.-L. Wang, On spectral approximations in elliptic geometries using Mathieu functions, Math. Comp. 78 (266) (2009) 815–844.

[42] Y. Shkolnisky, M. Tygert, V. Rokhlin, Approximation of bandlimited functions, Appl. Comput. Harmon. Anal. 21 (3) (2006) 413–420.

[43] F.J. Simons, F.A. Dahlen, M.A. Wieczorek, Spatiospectral concentration on a sphere, SIAM Rev. 48 (3) (2006) 504–536 (electronic).

[44] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty. IV. Extensions to many dimensions generalized prolate spheroidal functions, Bell System Tech. J. 43 (1964) 3009–3057.

[45] D. Slepian, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. I, Bell System Tech. J. 40 (1961) 43–63.

[46] D. Slepian, E. Sonnenblick, Eigenvalues associated with prolate spheroidal wave functions of zero order, Bell System Tech. J. 44 (1965) 1745–1759.

[47] G. Szegö, Orthogonal Polynomials, vol. 23, fourth edition, AMS Coll. Publ., 1975.

[48] M.A. Taylor, B.A. Wingate, A generalization of prolate spheroidal functions with more uniform resolution to the triangle, J. Engrg. Math. 56 (3) (2006) 221–235.

[49] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1988.

[50] G. Walter, Prolate spheroidal wavelets: translation, convolution, and differentiation made easy, J. Fourier Anal. Appl. 11 (1) (2005) 73–84.

[51] G. Walter, X.P. Shen, Wavelets based on prolate spheroidal wave functions, J. Fourier Anal. Appl. 10 (1) (2004) 1–26.

[52] G. Walter, T. Soleski, A new friendly method of computing prolate spheroidal wave functions and wavelets, Appl. Comput. Harmon. Anal. 19 (3) (2005) 432–443.

[53] L.-L. Wang, Analysis of spectral approximations using prolate spheroidal wave functions, Math. Comp., available online since September 2009.

[54] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1966.

[55] H. Xiao, Prolate spheroidal wave functions, quadrature, interpolation, and asymptotic formulae, PhD thesis, Yale University, 2001.

[56] H. Xiao, V. Rokhlin, High-frequency asymptotic expansions for certain prolate spheroidal wave functions, J. Fourier Anal. Appl. 9 (6) (2003) 575–596.

[57] H. Xiao, V. Rokhlin, N. Yarvin, Prolate spheroidal wavefunctions, quadrature and interpolation, Inverse Problems 17 (4) (2001) 805–838, Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[58] A.I. Zayed, A generalization of the prolate spheroidal wave functions, Proc. Amer. Math. Soc. 135 (7) (2007) 2193–2203 (electronic).

[59] J. Zhang, L.-L. Wang, Z. Rong, Spectral-element methods on quasi-uniform grids for nonlinear PDEs on a sphere, submitted for publication.