The World Around Us:
A Statistical View

Cheong Siew Ann

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The Millenium Prize Questions

- On 24 May 2000, the Clay Mathematics Institute announced a list of 7 important problems in mathematics, each prized at US$1,000,000:
  - P versus NP;
  - The Hodge Conjecture;
  - The Poincaré Conjecture;
  - The Riemann Hypothesis;
  - Yang-Mills Existence and Mass Gap;
  - Navier-Stokes Existence and Smoothness;
  - The Birch and Swinnerton-Dyer Conjecture.

The Sixth Problem . . .

- Navier-Stokes Equation of Fluid Dynamics: Prove the existence or non-existence of smooth solutions.

- Why the bounty?
  - Results on nonlinear PDEs notoriously difficult to obtain;
  - Importance of Navier-Stokes Equation in engineering.

- Implications to physics?

. . . and the Holy Grail of Physics

- Microscopic constituents of our world governed by Quantum Mechanics — the Schrödinger Equation:

$$\left[-\sum_{j}^{N}\frac{\hbar^2}{2m_j}\nabla_j^2+V(\mathbf{r}_1,\ldots,\mathbf{r}_N)\right]\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t). \tag{1}$$

- Macroscopic objects with low velocities relative to each other governed by Newton's Laws. For liquids and gases — the Navier-Stokes Equation:

$$\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\eta\nabla^2\mathbf{u}+\mathcal{F}. \tag{2}$$

- Feynman(?): Derive the Navier-Stokes Equation from the Schrödinger Equation for $\sim 10^{23}$ molecules.

A Classical Look at Liquids

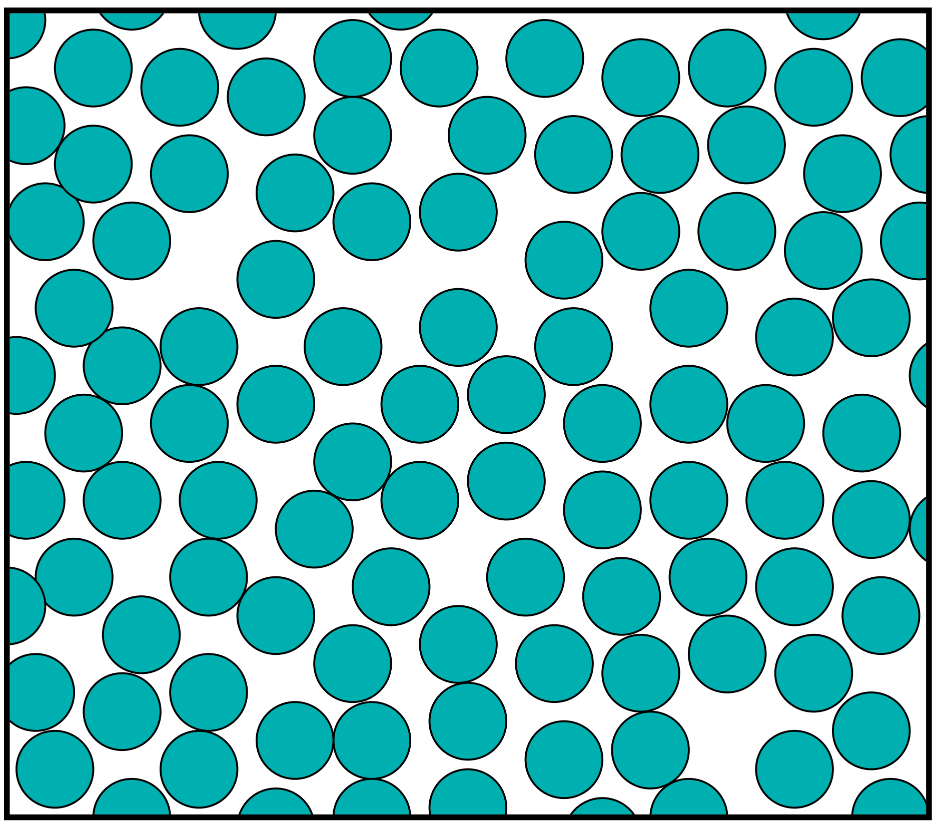
- Quantum problem of liquid hard, so try to appreciate the problem from a classical point of view first.

- Classically, liquid made up of a huge collection ($\sim 10^{23}$) of closely packed molecules, which interacts strongly with each other.

- Short range correlations, but no long range order $\Longrightarrow$ cannot treat as a weakly interacting gas, but also cannot treat as a highly ordered solid.

- macroscopic vs microscopic — macroscopic properties like density, pressure, temperature, . . . not well defined in this discrete microscopic picture.

Caricature of a Classical Liquid

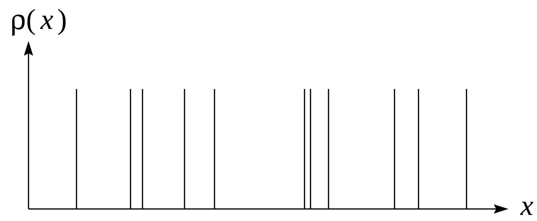


Defining Density

- Textbook definition is ρ = mass/volume. In a liquid want to define the density as a function of position $\mathbf{r} = (x, y, z)$.

- But mass concentrated at molecules $\Longrightarrow$ ρ high where the molecules are, but zero where there is no molecule.

- Density as a function of position has large and random fluctuations. For example, microscopic density function of 1-dimensional gas looks like



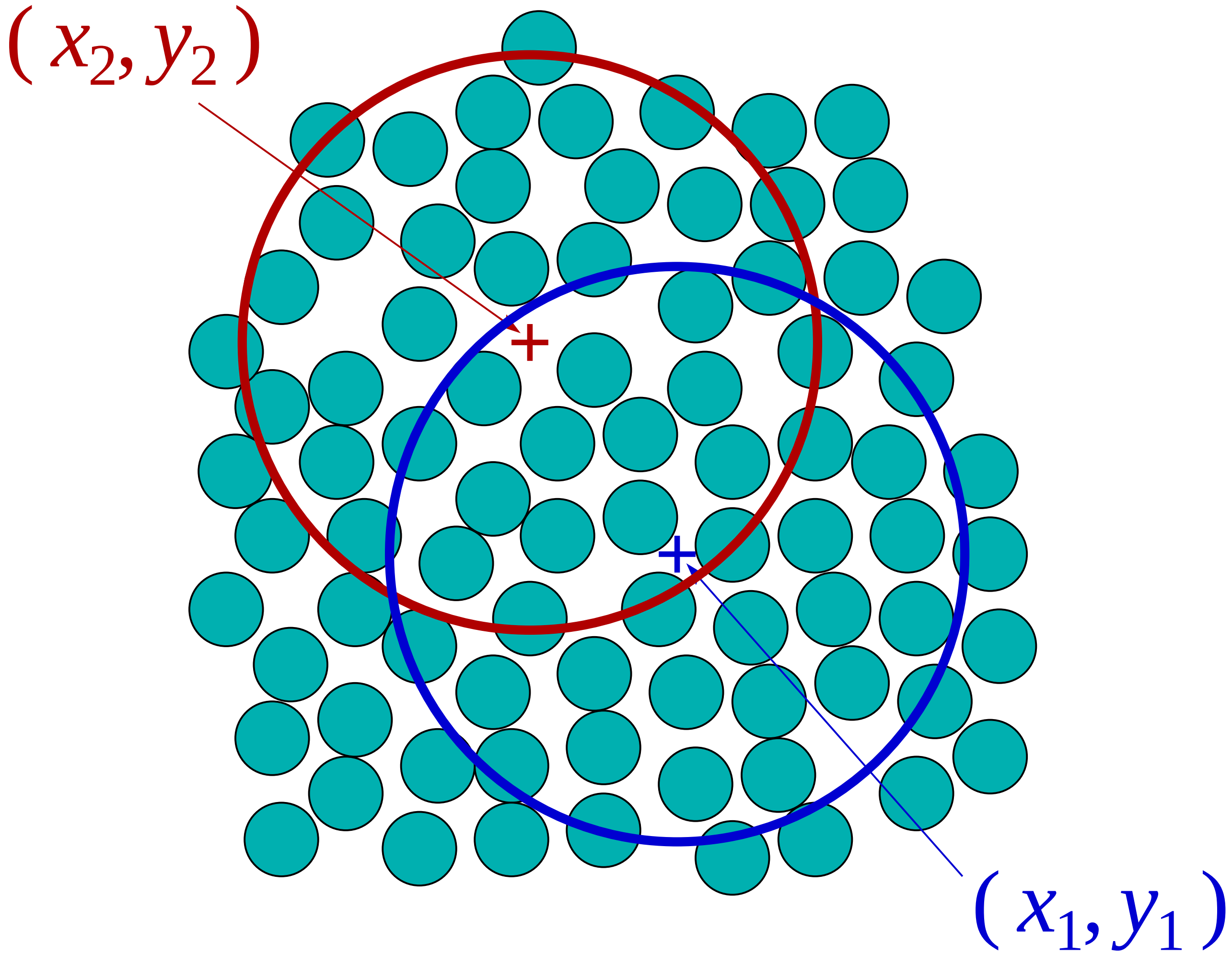
Difficult to handle such a density function.

Smoothing With Moving Averages

- Do not have such microscopic spatial and temporal resolutions in typical measurements of macroscopic quantities $\Longrightarrow$ $\rho(\mathbf{r}, t)$ actually some sort of averaged value over our limited resolution at **r** and t.

- Smooth the density function using moving average approach:
  - choose sampling sphere of radius R;
  - centre it at $\mathbf{r} = (x, y, z)$;
  - determine the total mass m inside the sampling sphere;
  - $\rho(\mathbf{r}, t) = m/V$, $V =$ volume of sampling sphere.

Caricature of Moving Average



A Smoother Density Function

- Equivalent to convolution of the original microscopic density function with a spherical window function.

- $\rho(\mathbf{r}, t)$ now much smoother, although there are still bumps when **r** is varied. This happens whenever our sampling sphere gains or lose one or more molecules. Size of bump, or fluctuations, depend on the distribution of positions of the molecules.

- For uniformly randomly distributed molecules,

$$\text{size of fluctuations} \propto R^2; \qquad \text{total mas enclosed} \propto R^3. \tag{3}$$

i.e. size of fluctuations in $\rho(\mathbf{r}, t) \propto 1/R$ as $R \to \infty$, i.e. fluctuations weighted down.

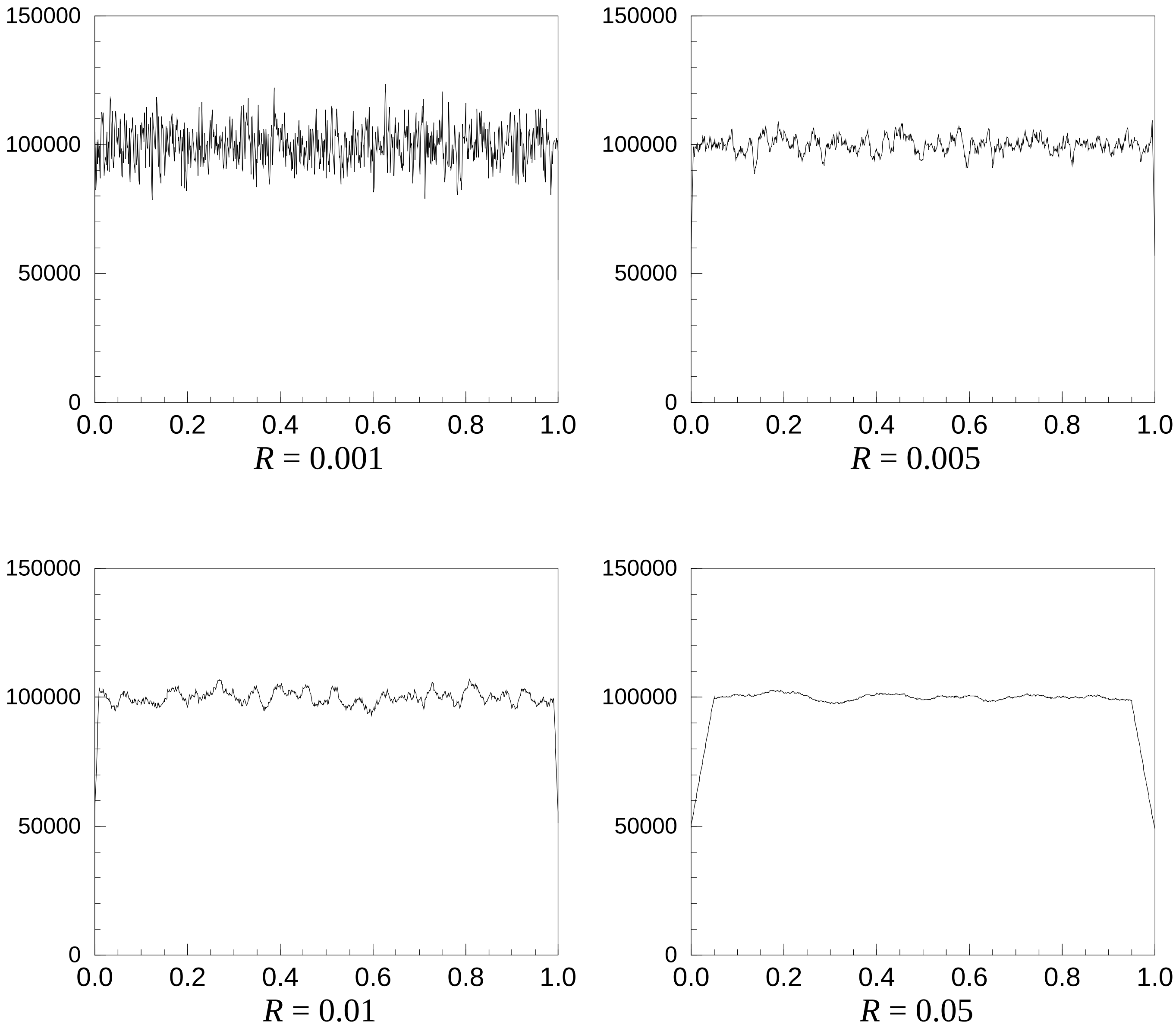
Remarks

- Spatial fluctuations decays as $1/R$, which is very slow.

- At each given sampling radius R, temporal fluctuations of $\rho(\mathbf{r}, t)$ dies off exponentially under thermal equilibrium conditions.

- If fluctuations decay exponentially with R, then beyond some radius R, the amplitude of the fluctuations of $\rho(\mathbf{r}, t)$ will become practically negligible.

- Not the case here, i.e. spatial fluctuations are supressed but still physically important.


150000
100000
50000
0
0.0
0.2
0.4
0.6
0.8
1.0
R = 0.001
R = 0.005
R = 0.01
R = 0.05

A Mean Field Description of Liquids

- Starting from classical picture of discrete molecules, arrive at the continuum description of liquid by means of averaging.

- Fluctuations suppressed to give mean field description.

- Quantum fluctuations if starting from quantum mechanics. But some method of averaging can be applied to yield mean field picture.

- In this sense, the Navier-Stokes Equation is the mean field description of fluids.

Breakdown of Mean Field Description

- We appear to have some recipe for arriving at a macroscopic picture starting from a disparate microscopic picture. So do we have our holy grail?

- We surmised that for a uniformly distributed fluid, the mean field density function $\rho(\mathbf{r}, t)$ is well defined and physically meaningful.

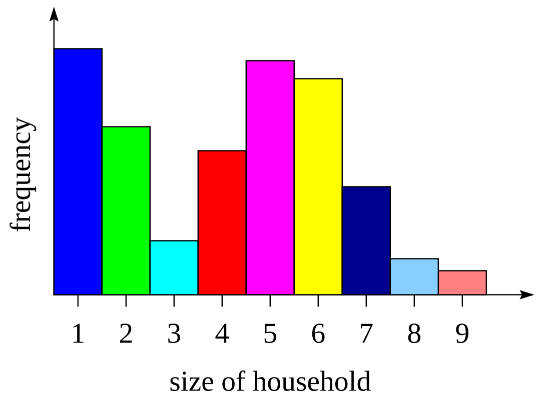
- In such situations, the average gives a good description of what goes on in the liquid. But such a mean field picture fails under the following circumstances:

Multi-Modal Distributions

- Example: A survey gives the average household size in US to be 2.5 persons.

- However, if pick US household at random, will find either a single person living alone, or big families with many kids. In fact, very few US households has 2 to 3 persons.

- If plot frequency distribution of household size, find that it looks like



- We say that such a distribution is bimodal, i.e. there are two modes in the distribution. In distributions with more than one mode, the mean is a very bad description of the system.

- In fact, even for unimodal distributions which are heavily skewed, the mean is already not representative.

Turbulence

- Mean field description for liquids also breaks down in turbulence.

- Reynolds number is defined as

$$\mathsf{Re} = \frac{\rho u d}{\eta}. \tag{4}$$

- At low Reynolds numbers, liquid flow is smooth and regular. If introduce small marker particle, will find it carried along by the liquid flow along a smooth curve.

- At high Reynolds numbers, smooth nature of liquid flow breaks down, and flow is characterized by eddy currents, or vortices of various sizes. Marker particle then found to follow very erratic path as liquid carries it along.

Phase Transitions

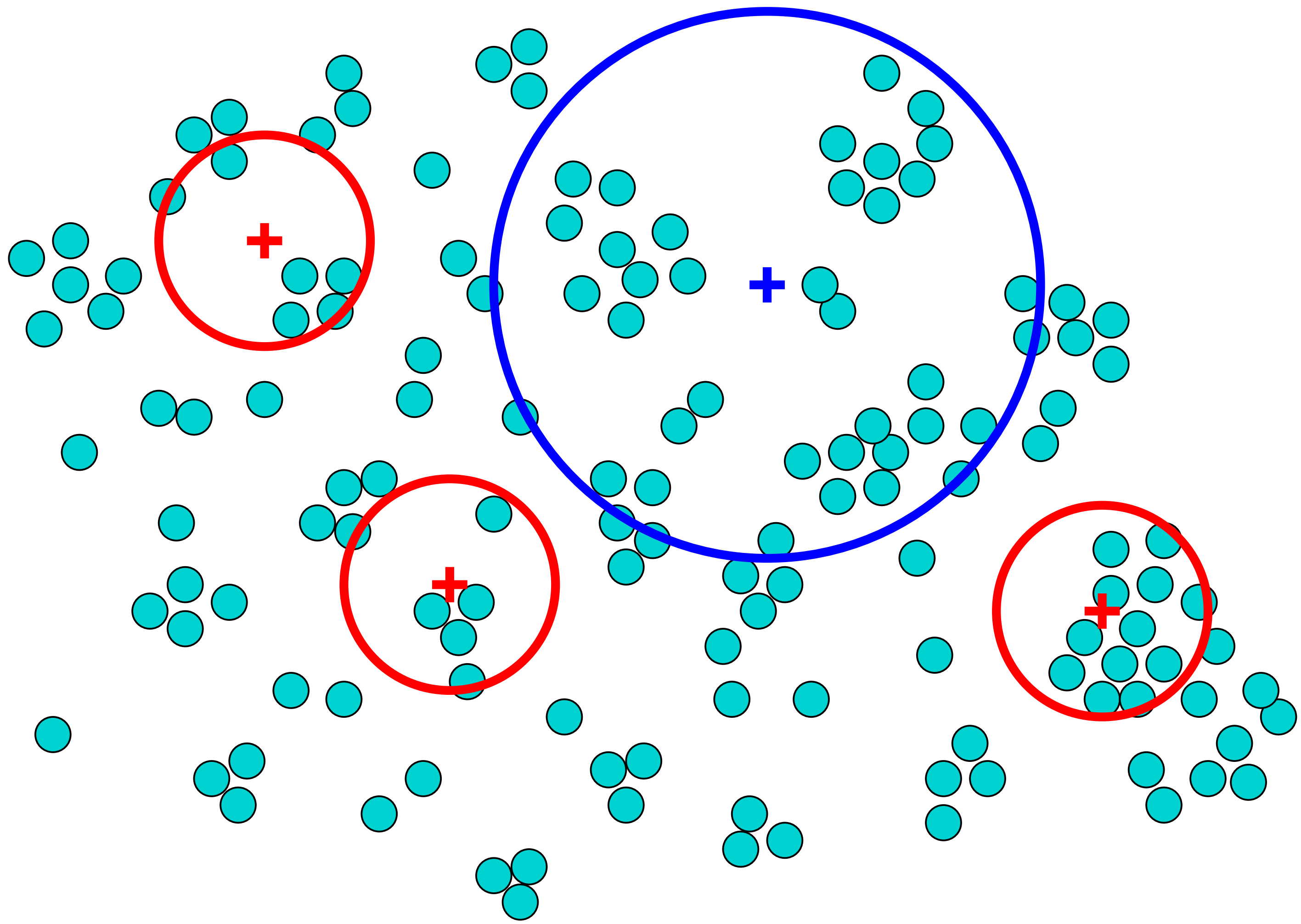
- At the liquid-to-gas phase transition, mixture of liquid and gaseous phases.

- Origin of strong density fluctuations somewhat like that of multimodal distributions — liquid and gaseous phase have different densities.

- Moving average method will always produce some number that can be interpreted as a density. But meaningless since the fluctuations is so large.

- In fact, at a phase transition, fluctuations of all scales equally important. Although most macroscopic properties can be computed, no longer provide an adequate description of the liquid. For example, how to explain macroscopically why there is a latent heat?

Caricature of a Liquid At Its Boiling Point



Scale Invariance and Renormalization

- Until now seen one of the major successes of statistical physics — that under ordinary circumstances, most physical systems admit a mean field description.

- However, seen when the mean field description fails to provide a representative picture of the system of interest — when fluctuations become so strong that its amplitude is greater than or comparable to the mean.

- In exploitation of wildest of fluctuations that statistical physics has its second major success: that of scale invariance and renormalization.

The Physics of Scale

- Imagine a superpowerful microscope which can see random molecular motion in a liquid.
- At highest magnification, the most striking pattern is the lack of thereof, as the molecules jiggle randomly around.
- Lower magnification, then start seeing a coordinated flow superimposed on the random motion.
- Even lower magnification, start seeing convection currents, but can no longer resolve the random molecular motion.
- No magnification, see the coordinated motion of convection currents in Rayleigh-Benard flow.

- In this example, magnification sets the length scale. The scale determines what kind of physics is dominant, and physics of the same system may look different at different scale.

Scale Invariance

- At phase transition, wild fluctuations of macroscopic quantities at all scales $\Longrightarrow$ no amount of fiddling with the shape and size of sampling sphere can smooth them out.

- Fluctuations statistically similar at all scales — scale invariance.

- For example, if choose a sampling sphere of radius R, get $\rho_R(\mathbf{r}, t)$. If then choose sampling sphere of radius λR, get $\rho_{\lambda R}(\mathbf{r}, t)$. Fluctuations in ρ_R and $\rho_{\lambda R}$ has same statistical properties.

- Such scale invariance more apparent in Ising systems, although generic of phase transitions.

- How to exploit scale invariance?

Power Laws in Statistical Physics

- Let x be a physical quantity, and let $p(x)$ be the distribution of x.

- If physics contained in x scale invariant, then if rescale x by λ, distribution for λx should look like distribution for x, i.e.

$$p(\lambda x) = f(\lambda)p(x). \tag{5}$$

- Solution to this functional equation is

$$p(x) = Ax^{\alpha}. \tag{6}$$

For physically meaningful distribution of x, $\alpha < 0$. This kind of distribution is called power law.

- α called scaling exponent or critical exponent.

The Renormalization Group

- Scaling exponents can be determined experimentally, but for some physical systems at their critical points, can be theoretically calculated using techniques collectively known as the Renormalization Group (RG).

- Idea behind RG simple:
  - start from fundamental scale of system;
  - calculate some probability distribution that is scale invariant at critical point;
  - Rescale system by λ and calculate the probability distribution again.

- Since probability distribution scale invariant at the critical point, equate the two distributions and obtain a RG Flow Equation as function of λ, and read off scaling exponent.

Marching Through the Hierarchy of Physics

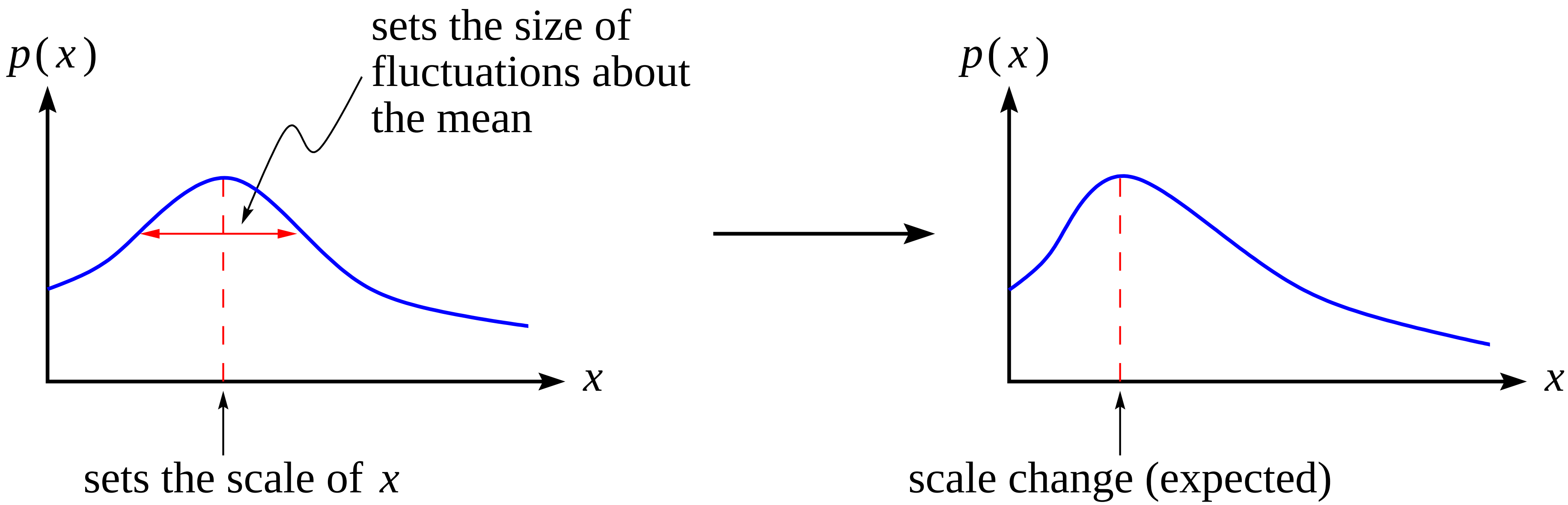
- Conquered both mean field description and fluctuations description? No, only the wildest fluctuations with RG.

- RG handles scale-free (scale invariant) systems well, but how to handle systems in which statistical fluctuations contain intrinsic length or time scale?

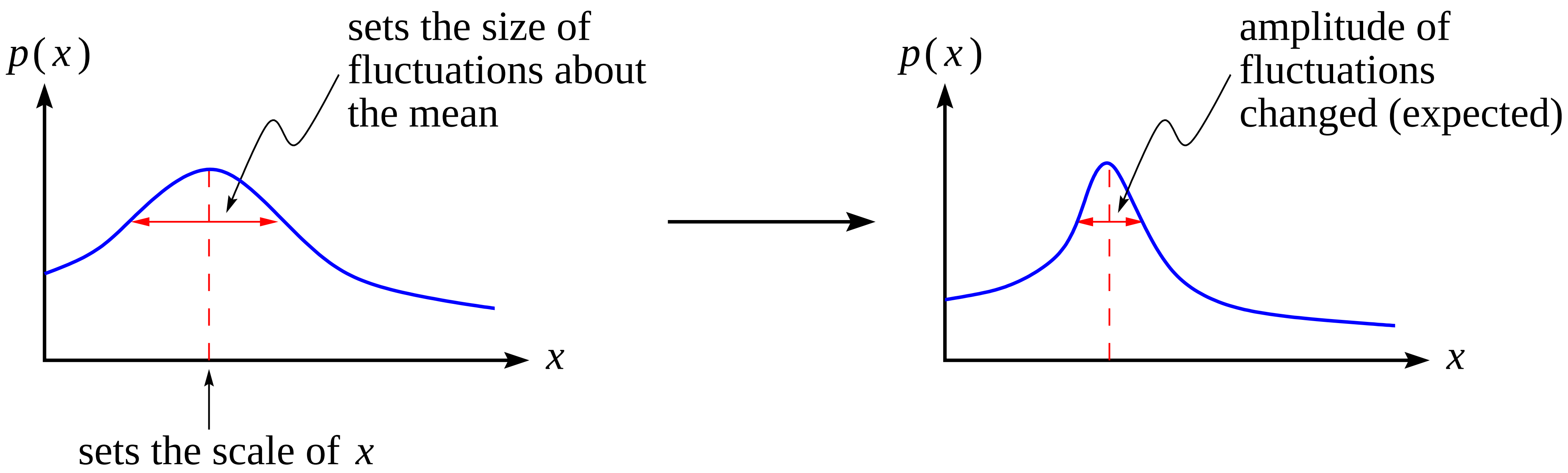
- Can use same RG machinery, except that now cannot assume that probability distribution remains invariant under scaling.

- Then track probability distribution as function of the scale. A few things can happen:

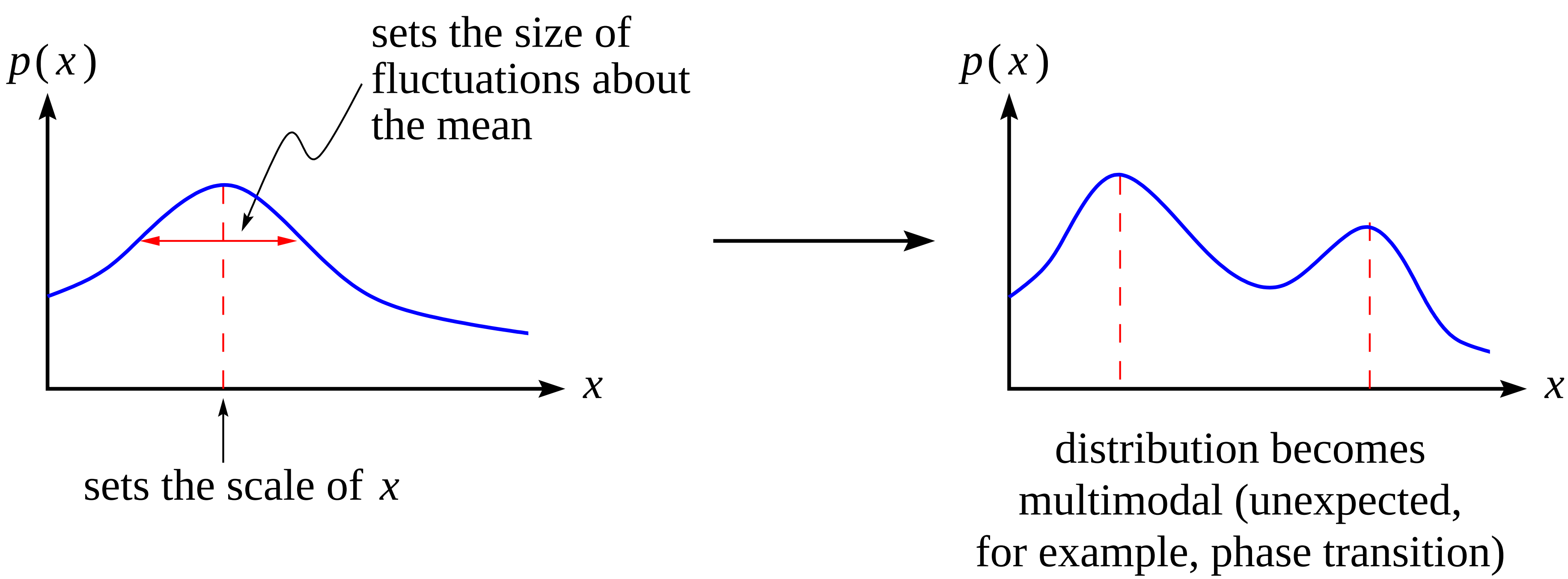
Change of Scale



Change in Amplitude of Fluctuations



Change in Form of Distribution



Mission Impossible . . . ?

- If we perform scaling and obtain a sequence of distributions $\{p_\lambda(x)\}$.

- At each scale λ, can then obtain the mean field description and see how the physics look like.

- But this is pure moot: we do not have a general prescription on how to do this in general yet.

Modeling, Mis-Modeling and Universal Results

- Problem of model mis-specification.

- Creative leap of imagination necessary from experimental observations to theoretical model — uniqueness and fidelity.

- 'Bad' if $p(x)$ and $q(x)$ are two slightly different distributions, but under scaling, give rise to vastly different sequence of physics!

- 'Good' if can make conclusions independent of details of model, i.e. want universal results.

Universality

- One of the most amazing discoveries in statistical physics — different physical systems can have the same critical exponents.

- The classes of physical systems having same critical exponents called universality classes. These share the same universal fluctuation statistics at their respective critical points.

- At critical points, the detailed differences between the physical systems are not important.

- Very important, since unlikely that we get all details correct.

- Universality is the most powerful statement we can make about physical systems under critical conditions.

Wanted: Universality

- Example: In social sciences, want to develop quantitative models.

- Common objections:
  - mis-specification of parameter values;
  - neglect of other parameters.

- Model produces qualitative agreement with observation. Convincing? Model ‘correct’?

- Universal conclusions: this universality class of models will produce this collection of qualitative and quantitative features, and no other universality class will.

What We Want to Learn From Universality Classes?

- Genericity — For example, say feature X produced by universality class U_X, and feature Y produced by universality class U_Y.

- U_X contains a broad spectrum of models. U_Y contains a narrow spectrum of models.

- Can argue that feature X is more generic than feature Y.

Summary

- Mean field description good when statistical fluctuations can be suppressed — but not a religion.
- Extreme fluctuations — scale invariance, power laws, scaling exponents and RG.
- When fluctuations not scale-free, then should renormalize the governing distribution of the model instead of mean field description. No clue how to do this yet.
- Universality and universality classes — can make conclusions that are independent of details of model.