Exact Ground States and Correlation Functions of Interacting Spinless Fermions on a Two-Legged Ladder

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Overview of Talk

- **Bosons and Fermions**: Brief review of Jordan-Wigner transformation.
- **Exact Ground State**: Trio of analytical maps relating 1D nearest-neighbor excluded and nearest-neighbor included periodic chains.
- **Correlation Functions**: Corresponding observables and the intervening-particle expansion.
- **Three Limiting Cases**: Extended Hubbard ladder of spinless fermions, overview of results, and zeroth-order ground-state phase diagram.
  - Strong correlated hopping limit.
  - Weak inter-leg hopping limit.
  - Strong inter-leg hopping limit.
- **Conclusions**.

The Jordan-Wigner Transformation

- P noninteracting spinless fermions on a 1D periodic chain of L sites,

$$H_c = -t \sum_{j=1}^{L} \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right).$$

- Ground state is a Fermi sea

$$|\Psi_F\rangle = \prod_{|k|<k_F} \tilde{c}_k^\dagger |0\rangle = \sum_{j_1<\cdots<j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)\, c_{j_1}^\dagger c_{j_2}^\dagger \cdots c_{j_P}^\dagger |0\rangle,$$

- Amplitude given by Slater determinant

$$\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P) = \frac{1}{L^{P/2}} \begin{vmatrix} e^{-\mathrm{i} k_1 j_1} & e^{-\mathrm{i} k_1 j_2} & \ldots & e^{-\mathrm{i} k_1 j_P} \\ e^{-\mathrm{i} k_2 j_1} & e^{-\mathrm{i} k_2 j_2} & \ldots & e^{-\mathrm{i} k_2 j_P} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\mathrm{i} k_P j_1} & e^{-\mathrm{i} k_P j_2} & \ldots & e^{-\mathrm{i} k_P j_P} \end{vmatrix}.$$

- Two-point function decays as power law, $\langle \Psi_F | c_i^\dagger c_j | \Psi_F \rangle \sim |i-j|^{-1}$.

The Jordan-Wigner Transformation

- P hard-core bosons on a 1D periodic chain of L sites,

$$H_b = -t \sum_j \left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + U \sum_j n_j (1 - n_j), \quad U \to \infty.$$

- Map to noninteracting spinless fermion using Jordan-Wigner transformation [P. Jordan and E. Wigner, *Z. Phys.* **47**, 631 (1928)],

$$b_i = \prod_{j<i} (1 - 2n_j)\, c_i = \prod_{j<i} (-1)^{n_j}\, c_i.$$

- Non-local operator $\prod_{j<i}(1 - 2n_j)$ called Jordan-Wigner string.
- Hard-core boson ground state

$$|\Psi\rangle = \sum_{j_1} \cdots \sum_{j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)|\; b_{j_1}^\dagger b_{j_2}^\dagger \cdots b_{j_P}^\dagger \,|0\rangle\,.$$

- Two-point function also decays as power law, $\langle \Psi | b_i^\dagger b_j | \Psi \rangle \sim |i - j|^{-1/2}$ [K. B. Efetov and A. I. Larkin, *Sov. Phys. JETP* **42**, 390 (1976)].

Nearest-Neighbor Inclusion & Exclusion

- 1D chain of hard-core bosons or spinless fermions with infinite nearest-neighbor repulsion

$$H_A = H_a + V \sum_j n_j n_{j+1}, \quad V \to \infty,$$

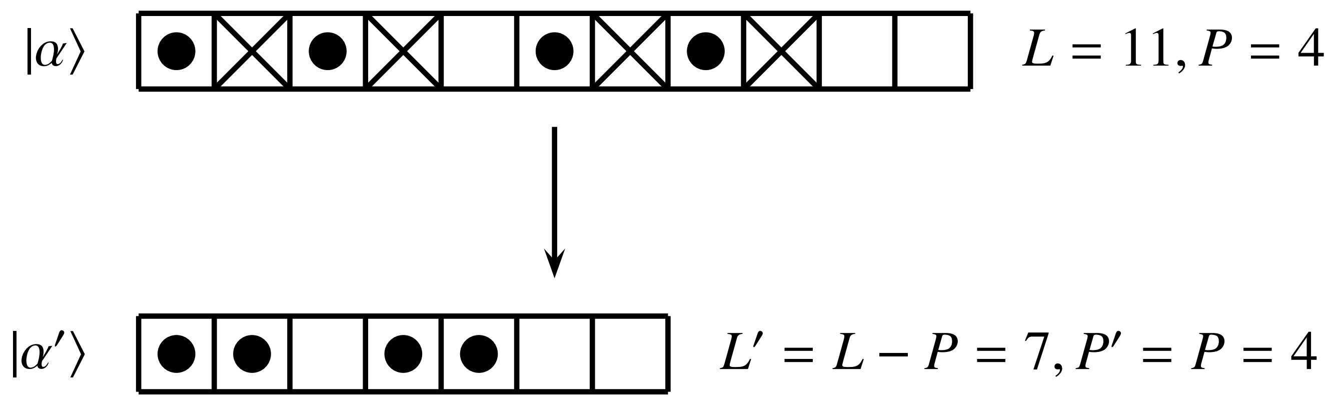
where $A = B$ (boson) or C (fermion), and $a = b$ (boson) or c (fermion).

- H_a allows nearest-neighbor occupation: Hilbert space $\mathscr{V}_a$ consists of nearest-neighbor included configurations.

- H_A forbids nearest-neighbor occupation: Hilbert space $\mathscr{V}_A$ consists of nearest-neighbor excluded configurations.

Configuration-to-Configuration Map

- Right exclusion map: nearest-neighbor excluded configuration to nearest-neighbor included configuration.



- Check that if $|\alpha\rangle \mapsto |\alpha'\rangle$ and $|\beta\rangle \mapsto |\beta'\rangle$, then $\langle\alpha|H_A|\beta\rangle = \langle\alpha'|H_a|\beta'\rangle$.

- Right exclusion map not one-to-one.

- Right inclusion map: nearest-neighbor included configuration to nearest-neighbor excluded configuration,

$$a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_P} |0\rangle \mapsto A^\dagger_{j_1} A^\dagger_{j_2+1} \cdots A^\dagger_{j_P+P-1} |0\rangle .$$

Bloch-State-to-Bloch-State Map

- Adopt closed-shell boundary conditions: P-fermion configuration incurs no sign change when translated across boundary. Treat bosons and fermions in same way.

- Translational invariance: define the Bloch states

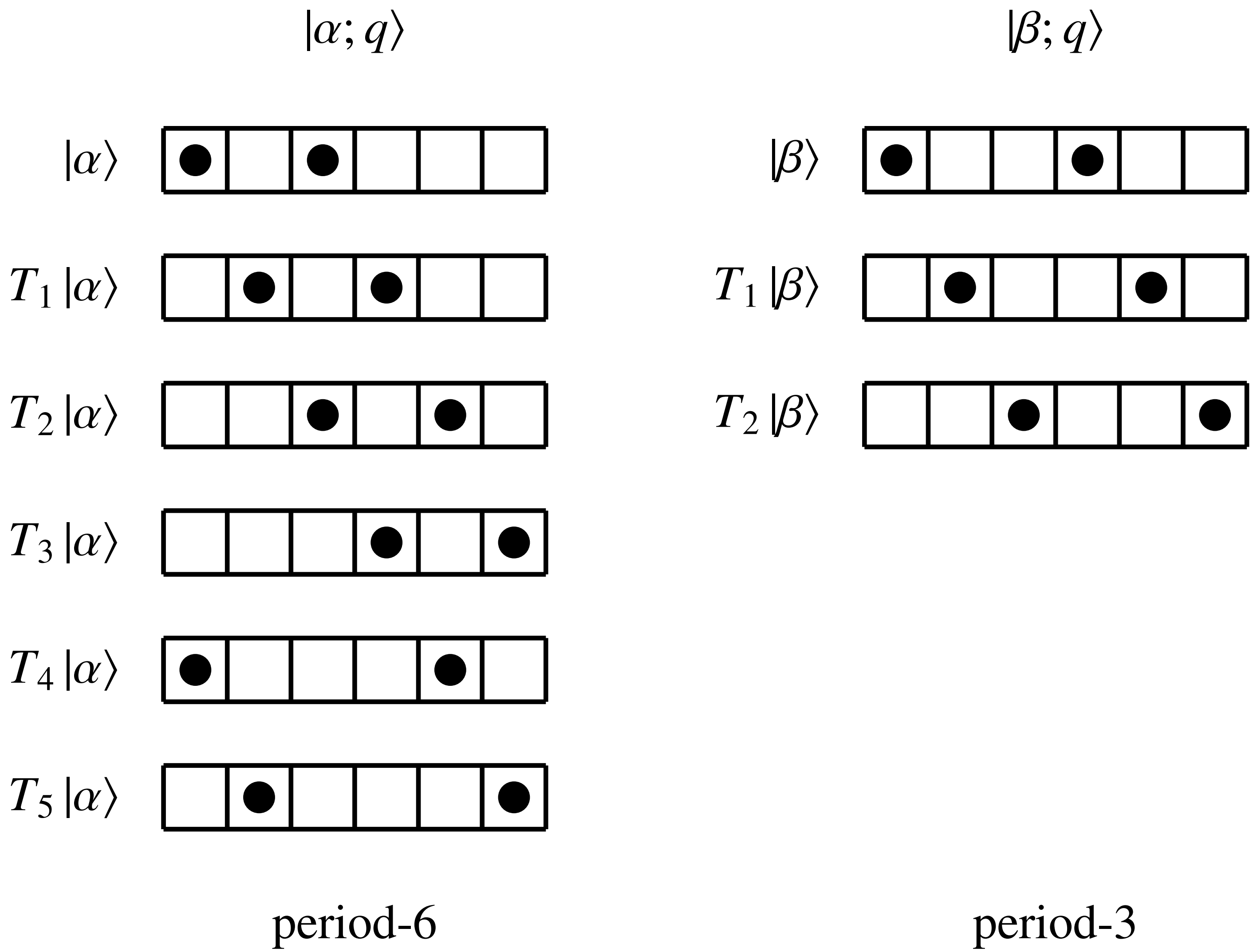
$$|\alpha; q\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{-\mathrm{i}\, q j}\, T_j\, |\alpha\rangle \,,$$

  where $|\alpha\rangle$ is generating P-particle nearest-neighbor excluded configuration, and T_j is translation operator.

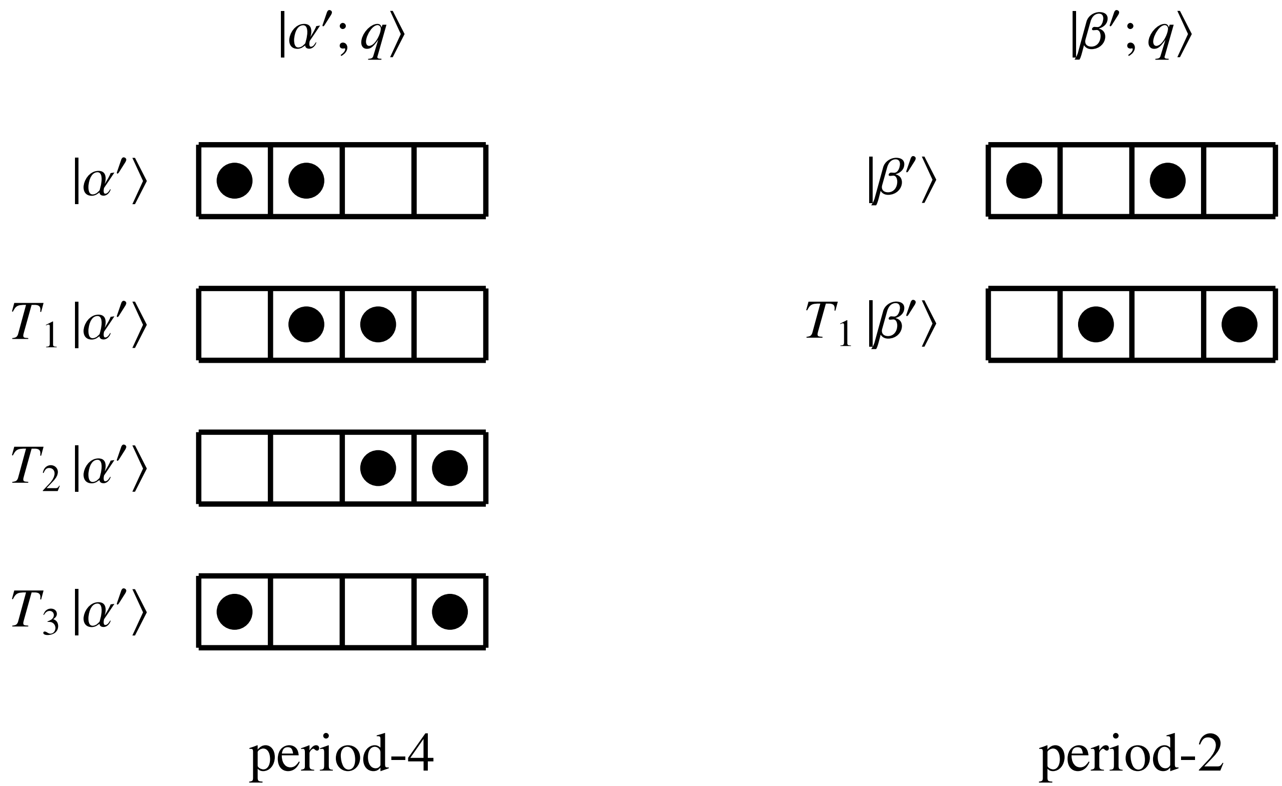
- Eigenstates of H_A have definite total linear momentum, and thus H_A block-diagonal in basis of Bloch states. Each diagonal block $H_A(q)$ characterized by total momentum wave vector q.

- Number of Bloch states = number of translationally inequivalent configurations.

Example: $L = 6,\ P = 2$



Example: $L' = 4,\ P = 2$



- For each q, two nearest-neighbor excluded Bloch states $|\alpha; q\rangle$ and $|\beta; q\rangle$.
- See that $|\alpha\rangle \mapsto |\alpha'\rangle$ and $|\beta\rangle \mapsto |\beta'\rangle$ under right-exclusion map.
- For each q', two nearest-neighbor included Bloch states $|\alpha'; q'\rangle$ and $|\beta'; q'\rangle$.
- Can we choose q and q' such that $\langle\alpha; q|H_A|\beta; q\rangle = \langle\alpha'; q'|H_a|\beta'; q'\rangle$?

Wave-Vector-To-Wave-Vector Map

- First note that nearest-neighbor excluded chain of length L maps to nearest-neighbor included chain of length $L' = L - P$.

- Allowed total-momentum wave vectors are

$$q = \frac{2\pi n}{L}, \quad q' = \frac{2\pi n'}{L'}, \quad n, n' \in \mathbb{Z}.$$

- Find that $\langle \alpha; q | H_A | \beta; q \rangle = \langle \alpha'; q' | H_a | \beta'; q' \rangle$ for all $|\alpha\rangle \mapsto |\alpha\rangle$ and $|\beta\rangle \mapsto |\beta'\rangle$ if we have

$$q = \frac{2\pi n}{L} \mapsto q' = \frac{2\pi n}{L'}, \quad n \in \mathbb{Z}.$$

- In case of $P = 1$, n simply the number of nodes in wave function.

Corollary of Combined Map

- $H_A(q)$ and $H_a(q')$ are identical as matrices. Same eigenvalues and eigenvectors.
- All nearest-neighbor excluded chain eigenstates can be written in terms of nearest-neighbor included chain eigenstates, and vice versa.
- In particular, if we know a nearest-neighbor included eigenstate with energy eigenvalue E' is

$$|\Psi'; q'\rangle = \sum_{j_1 < \cdots < j_P} \Psi'(q'; j_1, \ldots, j_P)\, a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_P} |0\rangle ,$$

then nearest-neighbor excluded eigenstate with the same energy eigenvalue $E = E'$ is

$$|\Psi; q\rangle = \sum_{j_1 < \cdots < j_P} \Psi'(q'; j_1, \ldots, j_P)\, A^\dagger_{j_1} A^\dagger_{j_2+1} \cdots A^\dagger_{j_P+P-1} |0\rangle ,$$

- Exact solution of nearest-neighbor excluded chain in terms of nearest-neighbor included chain!

Corresponding Observables

- Since $|\Psi'; q'\rangle$ and $|\Psi; q\rangle$ share the same amplitudes, want to cast problem of calculating $\langle O\rangle = \langle\Psi; q|O|\Psi; q\rangle$ in nearest-neighbor excluded chain as problem of calculating $\langle O'\rangle = \langle\Psi'; q'|O'|\Psi'; q'\rangle$ in nearest-neighbor included chain.

- Corresponding observables O and O' defined by their matrix elements between Bloch states,

$$\sqrt{l_\alpha l_\beta}\,\langle\alpha; q|O|\beta; q\rangle = \sqrt{l'_{\alpha'} l'_{\beta'}}\,\langle\alpha'; q'|O'|\beta'; q'\rangle\,,$$

  where l_α is period of $|\alpha\rangle$ and $l'_{\alpha'}$ is period of $|\alpha'\rangle$.

- Can check from right-exclusion map that $l'/l = \bar{n}'/\bar{n}$, where $\bar{n}$ is filling fraction in nearest-neighbor excluded chain, and $\bar{n}'$ is filling fraction in nearest-neighbor included chain.

- Expectation of corresponding observables related by

$$\langle O\rangle = \frac{\bar{n}}{\bar{n}'}\,\langle O'\rangle\,.$$

The Intervening-Particle Expansion

- Defining condition of corresponding observables stringent, satisfied by few observables. For generic observables, need to use intervening-particle expansion.

- Example: The intervening-particle expansion for $\langle A_i^\dagger A_{i+r}\rangle$ is

$$
\begin{aligned}
\langle A_i^\dagger A_{i+r}\rangle = {} & \langle A_i^\dagger(\mathbb{1} - N_{i+1})\cdots(\mathbb{1} - N_{i+r-1})A_{i+r}\rangle + \\
& \langle A_i^\dagger N_{i+1}\cdots(\mathbb{1} - N_{i+r-1})A_{i+r}\rangle + \cdots + \\
& \langle A_i^\dagger(\mathbb{1} - N_{i+1})\cdots N_{i+r-1}A_{i+r}\rangle + \\
& \langle A_i^\dagger N_{i+1}N_{i+2}\cdots(\mathbb{1} - N_{i+r-1})A_{i+r}\rangle + \cdots + \\
& \langle A_i^\dagger(\mathbb{1} - N_{i+1})\cdots N_{i+r-2}N_{i+r-1}A_{i+r}\rangle + \cdots + \\
& \langle A_i^\dagger N_{i+1}N_{i+2}\cdots N_{i+r-1}A_{i+r}\rangle\,.
\end{aligned}
$$

- Each term in expansion contains $p = 0, 1, \ldots, r$ intervening particles at fixed sites.

- Map each term $\langle A_i^\dagger O_p A_{i+r}\rangle$ to its corresponding expectation $\langle a_i^\dagger O'_p a_{i+r'}\rangle$, and then sum over $(\bar{n}/\bar{n}')\,\langle a_i^\dagger O'_p a_{i+r'}\rangle$ to get $\langle A_i^\dagger A_{i+r}\rangle$.

Rules for Corresponding Intervening-Particle Observables

- Nearest-neighbor exclusion: Drop terms $\langle A_i^\dagger O_p A_{i+r}\rangle$ in expansion if

$$A_j^\dagger A_{j+1}^\dagger, \quad A_j A_{j+1}, \quad A_j^\dagger N_{j+1}, \quad N_j A_{j+1}$$

  appear.

- Right-exclusion map: In the surviving terms, making the replacements

$$A_j^\dagger(\mathbb{1} - N_{j+1}) \mapsto a_j^\dagger, \quad A_j(\mathbb{1} - N_{j+1}) \mapsto a_j, \quad N_j(\mathbb{1} - N_{j+1}) \mapsto n_j.$$

- Re-indexing: Because right-exclusion map merges sites j and $j+1$, sites to right of $j+1$ must be re-indexed. For example,

$$N_j(\mathbb{1} - N_{j+1})N_{j+2} \mapsto n_j n_{j+1}.$$

  In general, site j on nearest-neighbor excluded chain becomes site $j-p$ on nearest-neighbor included chain if there are p particles between sites i and j (and including i).

Where We Are Right Now . . .

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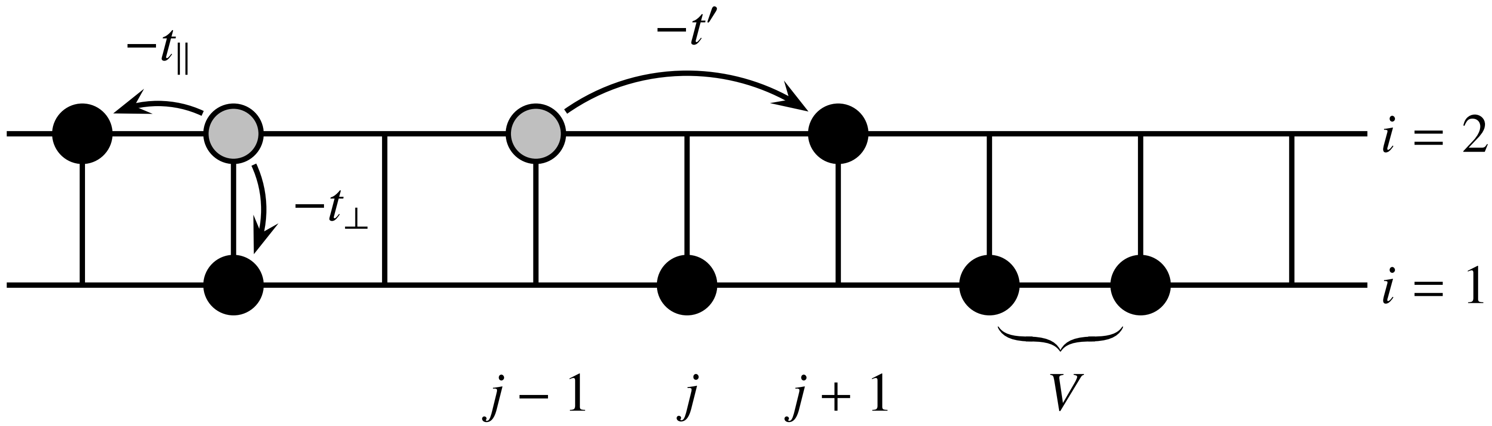
Extended Hubbard Ladder of Spinless Fermions

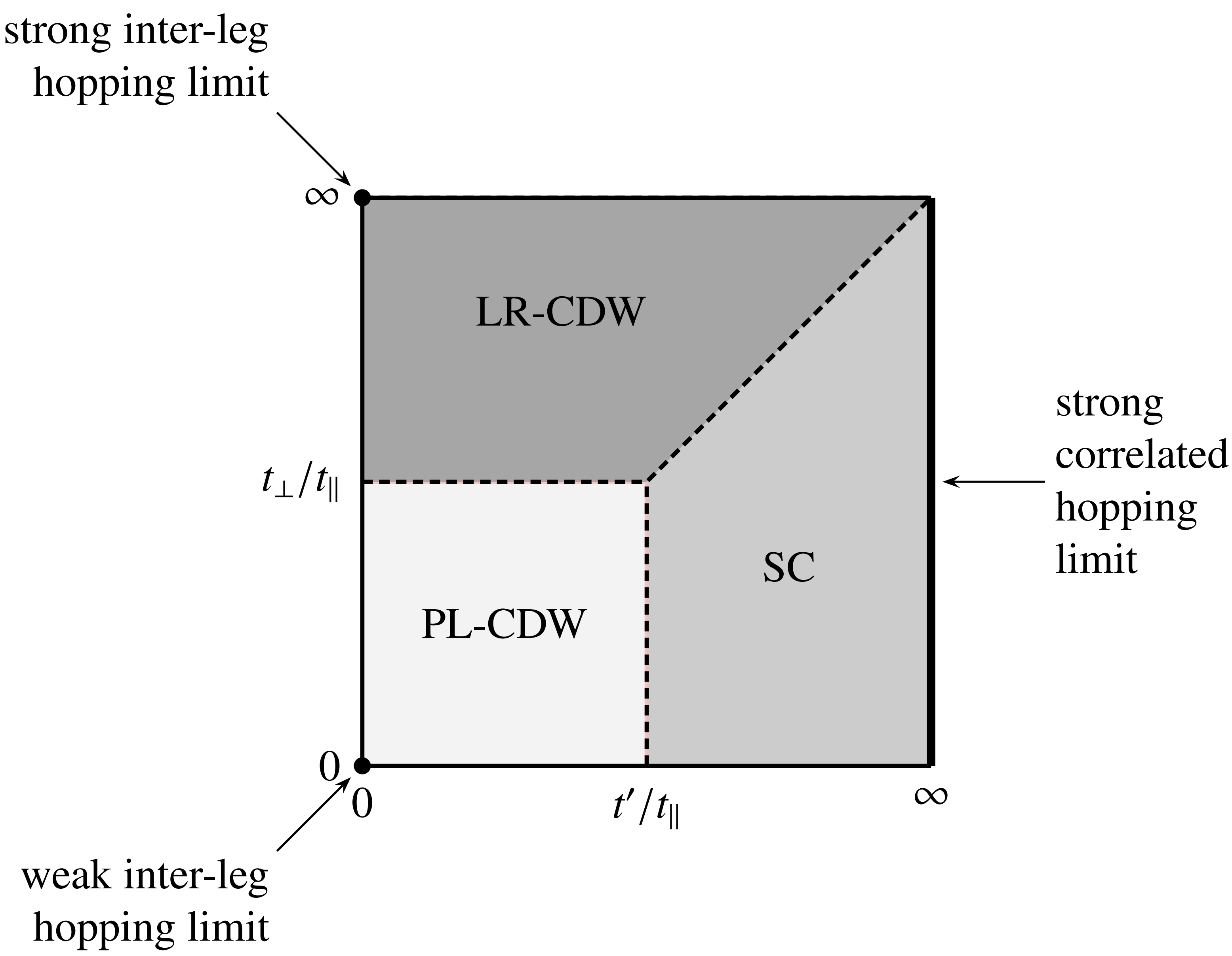


$$
\begin{aligned}
H_{t_{\|}t_{\perp}t'V} = &-t_{\|}\sum_i\sum_j\left(c^{\dagger}_{i,j}c_{i,j+1} + c^{\dagger}_{i,j+1}c_{i,j}\right) - t_{\perp}\sum_i\sum_j\left(c^{\dagger}_{i,j}c_{i+1,j} + c^{\dagger}_{i+1,j}c_{i,j}\right) \\
&- t'\sum_i\sum_j\left(c^{\dagger}_{i,j}n_{i+1,j+1}c_{i,j+2} + c^{\dagger}_{i,j+2}n_{i+1,j+1}c_{i,j}\right) \\
&- t'\sum_i\sum_j\left(c^{\dagger}_{i+1,j}n_{i,j+1}c_{i+1,j+2} + c^{\dagger}_{i+1,j+2}n_{i,j+1}c_{i+1,j}\right) \\
&+ V\sum_i\sum_j n_{i,j}n_{i,j+1} + V\sum_i\sum_j n_{i,j}n_{i+1,j}, \qquad V \to \infty.
\end{aligned}
$$

Overview of Three Limiting Cases

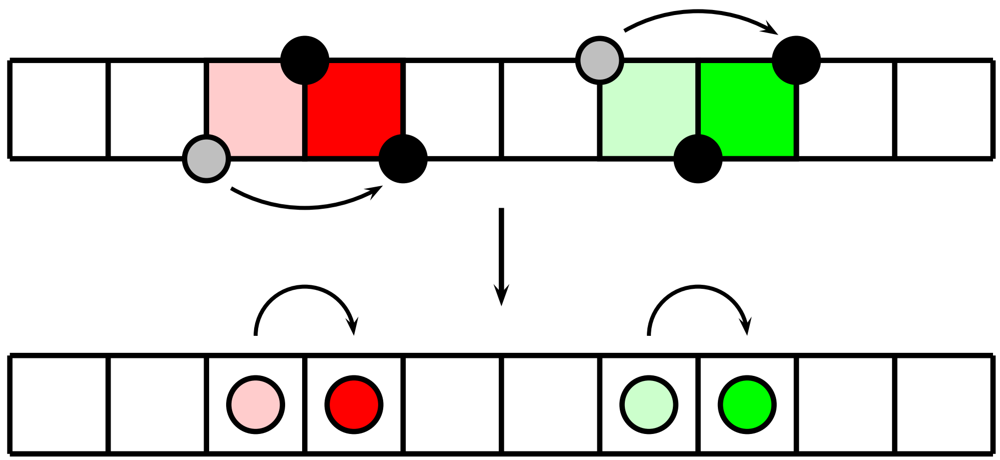
- Strong correlated-hopping limit, $t' \gg t_{\|}, t_{\perp}$:
  - universal SC power-law correlations dominate over non-universal hard-core-boson CDW power-law correlations at large distances.
  - FL correlations decay exponentially.
- Weak inter-leg hopping limit, $t_{\perp} \ll t_{\|}$, $t' = 0$:
  - universal CDW power-law correlations dominate over universal SC power-law correlations at large distances.
  - FL correlations decay exponentially.
- Strong inter-leg hopping limit, $t_{\perp} \gg t_{\|}$, $t' = 0$:
  - True long-range CDW when $\bar{n}_2 = \frac{1}{4}$.
  - Phase separation for $\bar{n}_2 > \frac{1}{4}$.
  - For $\bar{n}_2 < \frac{1}{4}$, universal SC power-law correlations dominate universal FL and CDW power-law correlations at large distances.

Zeroth-Order Phase Diagram



Strong Correlated Hopping Limit

- When $t' \gg t_{\|}, t_{\perp}$, ladder spinless fermions form well-defined pairs: 1D problem of interacting hard-core bosons.
- Two flavors of interacting hard-core bosons. Call them even and odd, or red (R) and green (G). Flavor conserved as fermion pair correlated-hops.

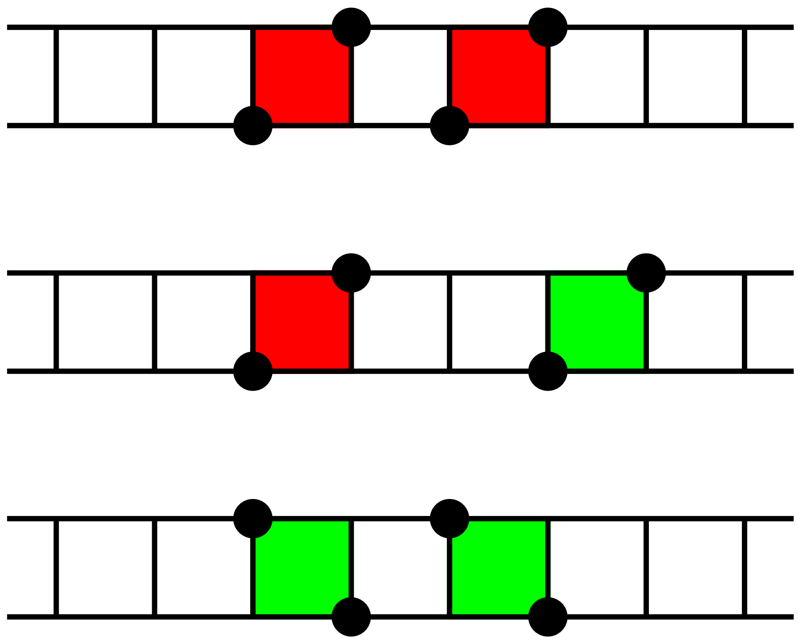


- Bound-pair-to-hard-core boson map:

$$B_j^\dagger = \begin{cases} c_{1,j}^\dagger c_{2,j+1}^\dagger, & j \text{ even;} \\ c_{1,j+1}^\dagger c_{2,j}^\dagger, & j \text{ odd,} \end{cases} \qquad B_j^\dagger = \begin{cases} c_{1,j+1}^\dagger c_{2,j}^\dagger, & j \text{ even;} \\ c_{1,j}^\dagger c_{2,j+1}^\dagger, & j \text{ odd.} \end{cases}$$

Strong Correlated Hopping Limit

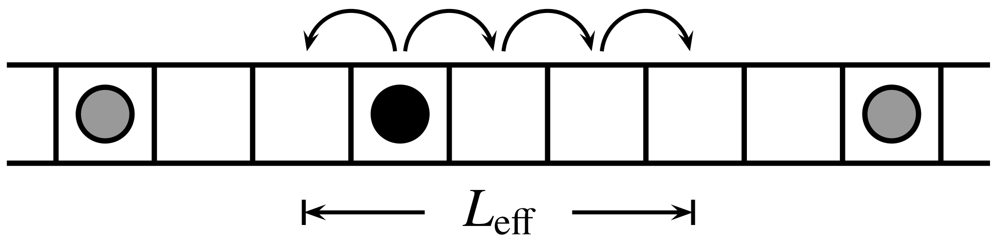
- Hard-core boson of each flavor can come within two sites of another hard-core boson of the same flavor, but can only come within three sites of a hard-core boson of different flavor. Hard-core bosons cannot exchange positions.



- For $2P$ spinless fermions on ladder of length L, Hilbert space breaks up into sectors of immutable flavor sequences. Example: For $P = 4$, the distinct flavor sequences are $RRRR$, $RRRG$, $RRGG$, $RGRG$, $RGGG$, and $GGGG$.

Kinetic Energy Argument

- Each hard-core boson confined to hop within interval of chain between the two hard-core bosons closest to it: particle-in-a-box problem!



- At given filling fraction $\bar{n}$,
  - L_{eff} larger if R particle bounded by R particles, and G particle bounded by G particles.
  - L_{eff} smaller if R particle bounded by G particles, or G particle bound by R particles.
  - kinetic energy of bound particle lowest if bound by particles of the same flavor.

- Two-fold-degenerate ground state for $2P$ spinless fermions: P R bound pairs or P G bound pairs. Ground-state wave functions of each can be mapped to ground-state wave function of P noninteracting spinless fermions.

Ground-State Wave Functions

- Start with ground-state wave function of P noninteracting spinless fermions on periodic chain of length $L' = L - P$,

$$|\Psi_F\rangle = \sum_{j_1 < \cdots < j_P} \Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)\, c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_P} |0\rangle ,$$

where $k_1, \ldots, k_P$ are the P occupied single-particle wave vectors.

- Use Jordan-Wigner map to get ground-state wave function of P nearest-neighbor included hard-core bosons on periodic chain of length $L' = L - P$,

$$|\Psi_b\rangle = \sum_{j_1 < \cdots < j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)|\, b^\dagger_{j_1} b^\dagger_{j_2} \cdots b^\dagger_{j_P} |0\rangle ,$$

- Use right-inclusion map to get ground-state wave function of P nearest-neighbor excluded hard-core bosons on periodic chain of length L,

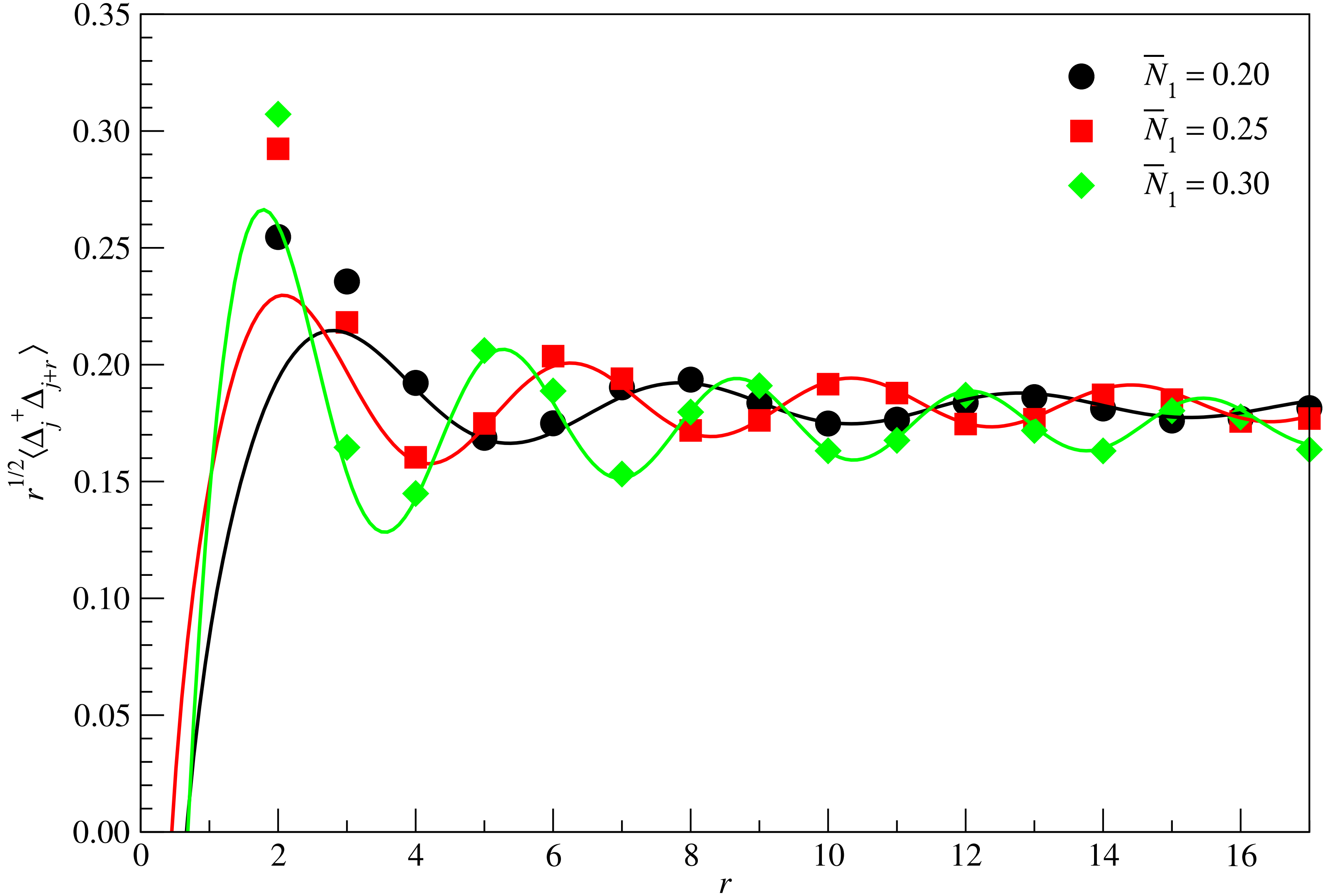
$$|\Psi_B\rangle = \sum_{j_1 < \cdots < j_P} |\Psi_F(k_1, \ldots, k_P; j_1, \ldots, j_P)|\, B^\dagger_{j_1} B^\dagger_{j_2+1} \cdots B^\dagger_{j_P+P-1} |0\rangle ,$$

- Use bound-pair-to-hard-core-boson map to get ground-state wave function of P (R or G) bound pairs on ladder of length L.

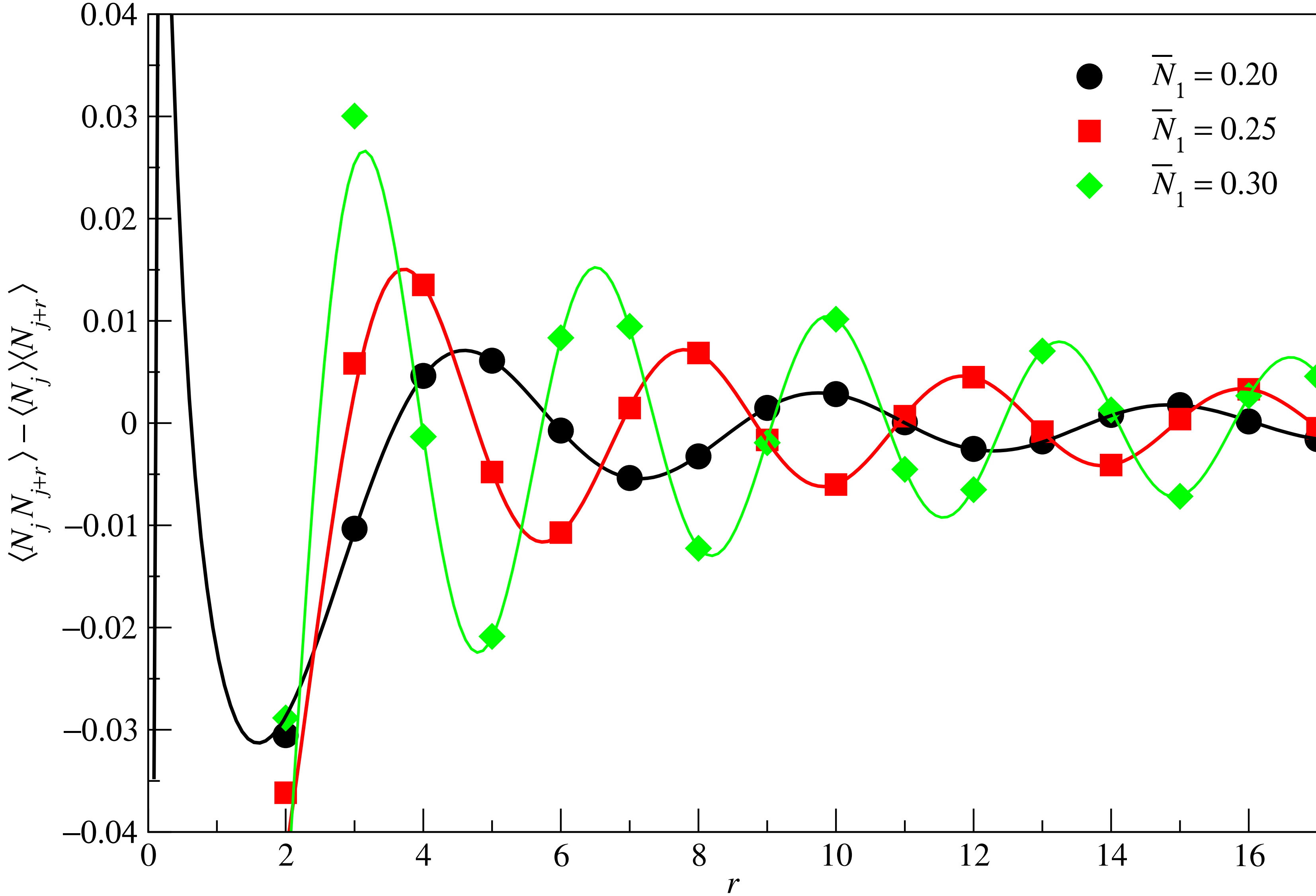
Correlation Functions

- Only simple to calculate correlation functions which can be written in terms of B_j and $B_j^\dagger$.
  - SC correlations $\langle B_i^\dagger B_{i+r} \rangle$.
  - CDW-π correlations $\langle B_i^\dagger B_i B_{i+r}^\dagger B_{i+r} \rangle$.
- Correlation functions not readily expressible in terms of B_j and $B_j^\dagger$ difficult to calculate.
  - FL correlation $\langle c_{i,j}^\dagger c_{i',j+r} \rangle$, understood using semi-quantitative arguments.
  - CDW-σ correlations $\langle c_{i,j}^\dagger c_{i,j} c_{i',j+r}^\dagger c_{i',j+r} \rangle$.
- Numerically, summing the intervening-particle expansion for correlation functions involve summing over various minors of an $r \times r$ matrix. Without acceleration schemes, only feasible up to separations of $r \approx 20$.
- Correlation exponents, wave vectors, amplitudes and phase shifts obtained through nonlinear curve fitting.

SC Correlations

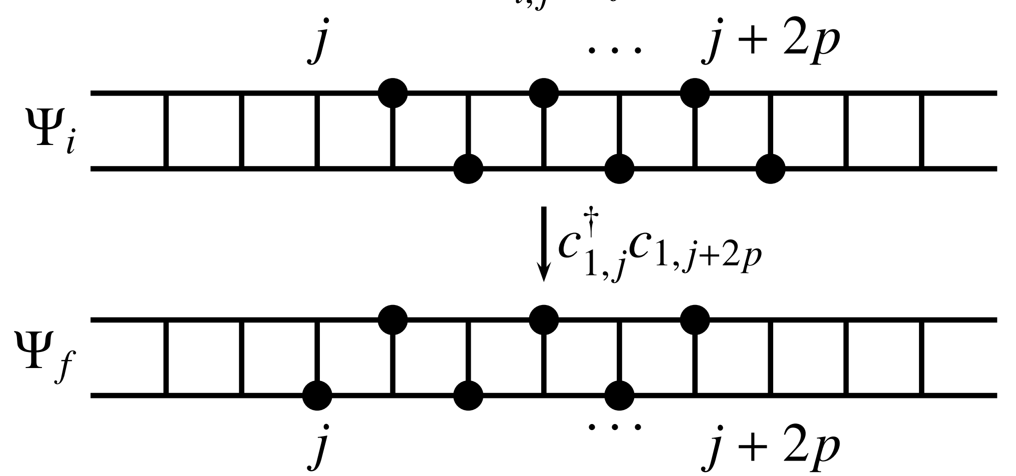


CDW-π Correlations



FL Correlations

- Configurations containing unpaired spinless fermions cannot occur in ground state.
- FL correlations $\langle c^\dagger_{i,j} c_{i',j+r} \rangle$ nonzero only when r even.
- For $r = 2p$, only compact p-bound-pair configurations with one end at j and the other end at $j + r$ contribute to $\langle c^\dagger_{i,j} c_{i',j+r} \rangle$.



- $\langle c^\dagger_{i,j} c_{i',j+r} \rangle$ proportional to probability of finding compact p-bound-pair cluster in ground state.
- Compact p-bound-pair cluster $\mapsto$ compact p-hard-core-boson cluster $\mapsto$ compact p-noninteracting-spinless-fermion cluster.

FL Correlations

- From SAC and C. L. Henley, Phys. Rev. B **69**, 075112 (2004), know that probability of fully-occupied p-site cluster in 1D Fermi sea is

$$\det G_C(p) = \prod_{l=1}^{p} \lambda_l = \prod_{l=1}^{p} \frac{1}{e^{\varphi_l} + 1},$$

where λ_l are eigenvalues of the cluster Green-function matrix $G_C(p)$, and φ_l are the single-particle pseudo-energies of the cluster density matrix ρ_C.

- For $p \gg 1$, know that

$$\det G_C(p) \approx \exp\left(-p \int_0^{1-\bar{n}'} f(\bar{n}', x)\, dx\right),$$

i.e. FL correlations decay exponentially for large r, with $\bar{n}$-dependent correlation length ($\bar{n}'$ is filling fraction of nearest-neighbor included chain).

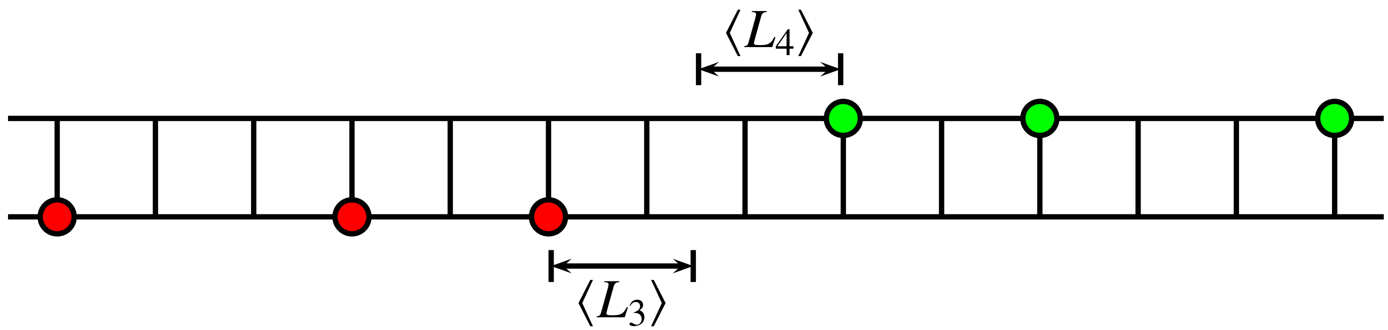
Summary of Correlation Exponents

limit	correlation function	correlation exponent	wave vector
$t' \gg t_\parallel, t_\perp$	CDW-π	$\frac{1}{2} + \frac{5}{2}\left(\frac{1}{2} - \bar{N}_1\right)$	$2k_F$
		2	0
	SC	$\frac{1}{2}$	0
		$\frac{3}{2} \to \frac{1}{2}$	$2k_F$

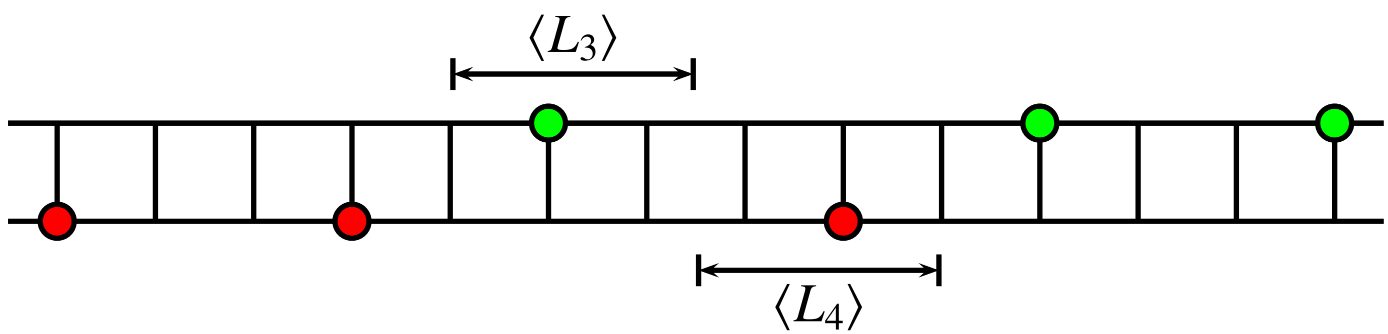
Weak Inter-Leg Hopping Limit

- When $t_\perp \to 0$ and $t' = 0$, the two legs of ladder coupled only by infinite nearest-neighbor repulsion.
- Each spinless fermion carries permanent leg index i.
- Spinless fermion cannot move past each other, even if they are on different legs (because of infinite nearest-neighbor repulsion).
- For P spinless fermions on ladder of length L, Hilbert space breaks up into sectors of immutable leg indices. Example: For $P = 4$, the distinct leg-index sequences are 1111, 1112, 1122, 1212, 1222, and 2222.
- Again use kinetic energy argument to determine structure of ground state:
  - Compare locally the sequences $\{\cdots 111222 \cdots\}$ and $\{\cdots 112122 \cdots\}$, find that third and fourth particles in $\{\cdots 112122 \cdots\}$ have longer intervals to hop around, compared to their counterparts in $\{\cdots 111222 \cdots\}$.

Weak Inter-Leg Hopping Limit



$\{\cdots 111222 \cdots\}$



$\{\cdots 112122 \cdots\}$

- Kinetic energies of particles forming leg-index domain wall lower.
- Overall ground state must therefore have as many domain walls as possible, i.e. sequence must be $\{\cdots 121212 \cdots\}$ or $\{\cdots 212121 \cdots\}$.
- Two-fold-degenerate staggered ground state.

Ground-State Wave Functions

- Again, start with ground-state wave function of P noninteracting spinless fermions on periodic chain of length $L' = L$,

$$|\Psi_F\rangle = \sum_{j_1<\cdots<j_P} \Psi_F(k_1,\ldots,k_P; j_1,\ldots,j_P)\, c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_P} |0\rangle\,,$$

  where $k_1,\ldots,k_P$ are the P occupied single-particle wave vectors. Infinite nearest-neighbor repulsion between different legs do not result in need to exclude sites.

- Without loss of generality, assume P even. Then two-fold-degenerate staggered ground-state wave functions are

$$|\Psi_\pm\rangle = \sum_{j_1<\cdots<j_P} \Psi_F(k_1,\ldots,k_P; j_1,\ldots,j_P) \times \frac{1}{\sqrt{2}}\left(c^\dagger_{1,j_1} c^\dagger_{2,j_2} \cdots c^\dagger_{1,j_{P-1}} c^\dagger_{2,j_P} \pm c^\dagger_{2,j_1} c^\dagger_{1,j_2} \cdots c^\dagger_{2,j_{P-1}} c^\dagger_{1,j_P}\right) |0\rangle\,.$$

  $|\Psi_+\rangle$ symmetric with respect to reflection about ladder axis, while $|\Psi_-\rangle$ antisymmetric with respect to reflection about ladder axis.

- Note that ladder with filling fraction $\bar{n}_2$ maps onto chain of filling fraction $\bar{n}_1 = 2\bar{n}_2$.

Correlation Functions

- CDW+ correlations

$$\langle n_{1,j} n_{1,j+r} \rangle + \langle n_{1,j} n_{2,j+r} \rangle \,,$$
$$\langle n_{2,j} n_{1,j+r} \rangle + \langle n_{2,j} n_{2,j+r} \rangle$$

  both equal $\frac{1}{2} \langle \Psi_F | n_j n_{j+r} | \Psi_F \rangle$, the CDW correlation in 1D Fermi sea.

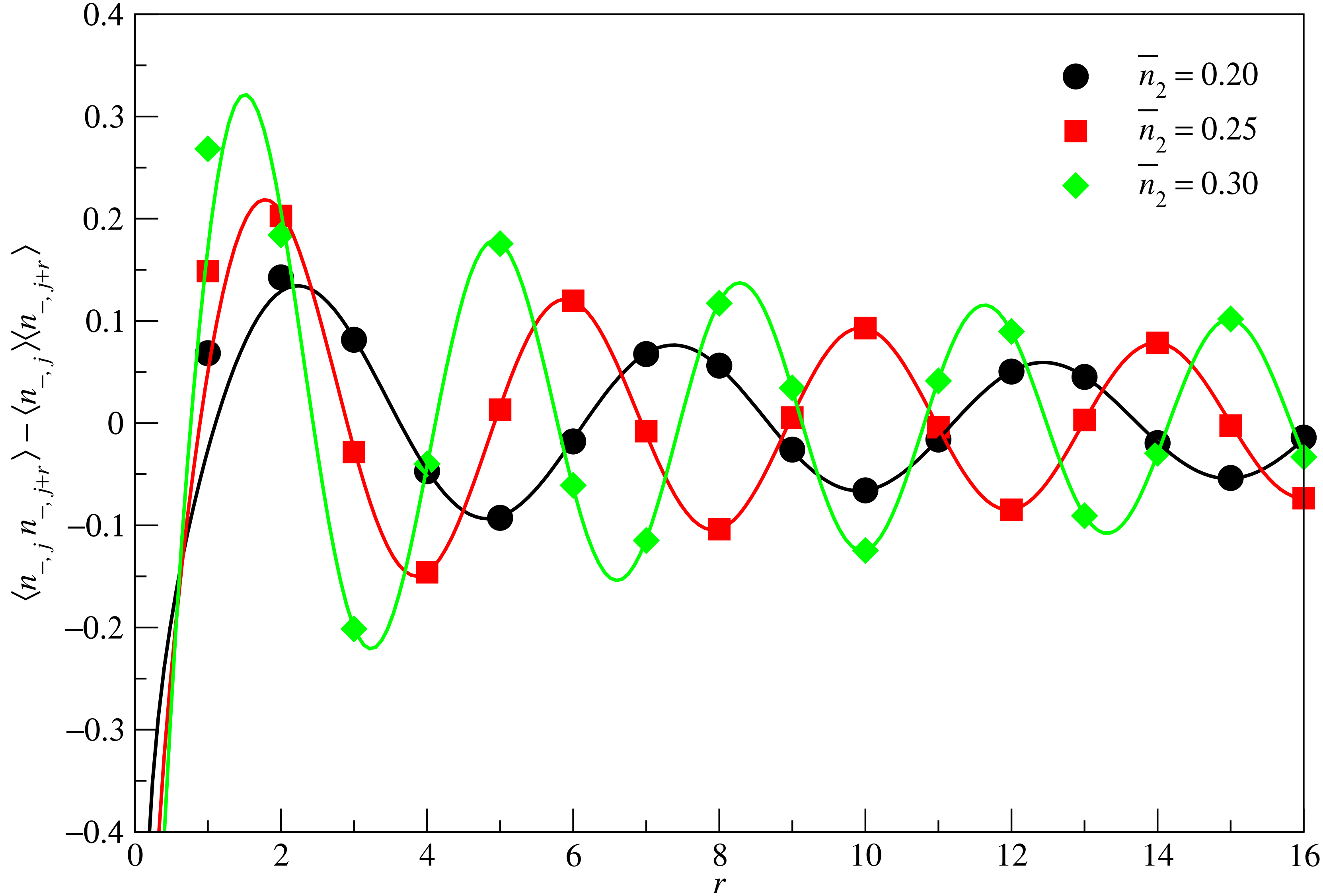
- SC+ correlations

$$\langle c^\dagger_{2,j+1} c^\dagger_{1,j} c_{1,j+r} c_{2,j+r+1} \rangle + \langle c^\dagger_{2,j+1} c^\dagger_{1,j} c_{2,j+r} c_{1,j+r+1} \rangle \,,$$
$$\langle c^\dagger_{1,j+1} c^\dagger_{2,j} c_{1,j+r} c_{2,j+r+1} \rangle + \langle c^\dagger_{1,j+1} c^\dagger_{2,j} c_{2,j+r} c_{1,j+r+1} \rangle$$

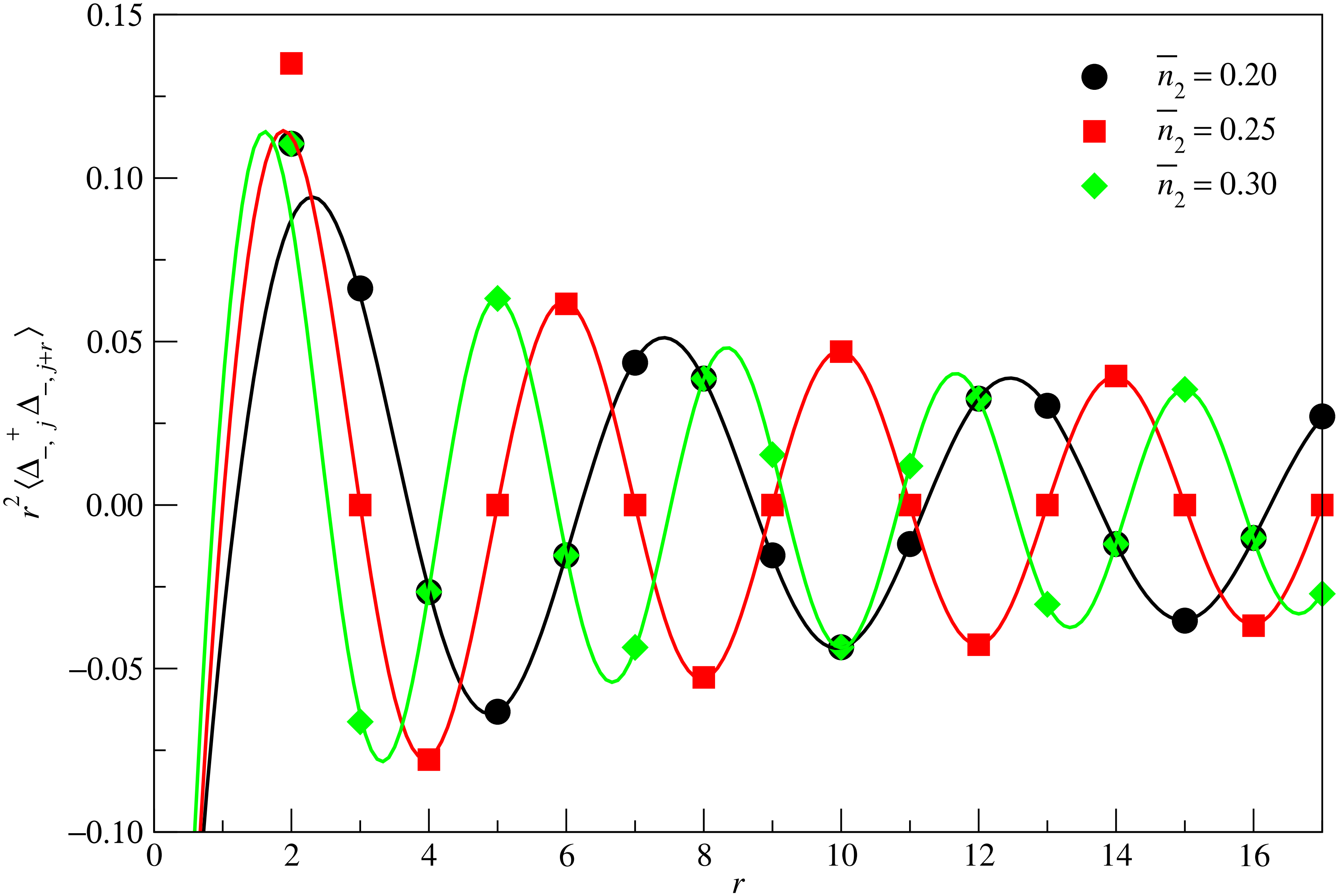
  both equal $\frac{1}{2} \langle c^\dagger_{j+1} c^\dagger_j c_{j+r} c_{j+r+1} \rangle$, the SC correlation in 1D Fermi sea.

- CDW− and SC− correlations need to calculate numerically.
- Staggered FL correlations $\langle c^\dagger_{1,j} c_{2,j+r} \rangle = 0 = \langle c^\dagger_{2,j} c_{1,j+r} \rangle$ vanish identically.
- FL correlations $\langle c^\dagger_{1,j} c_{1,j+r} \rangle$ and $\langle c^\dagger_{2,j} c_{2,j+r} \rangle$ decay exponentially with r, understood using semi-quantitative arguments.

CDW– Correlations

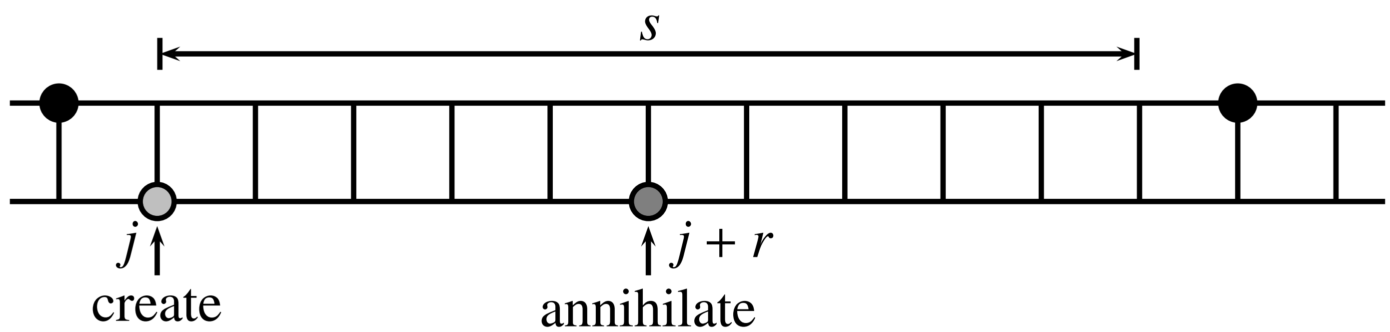


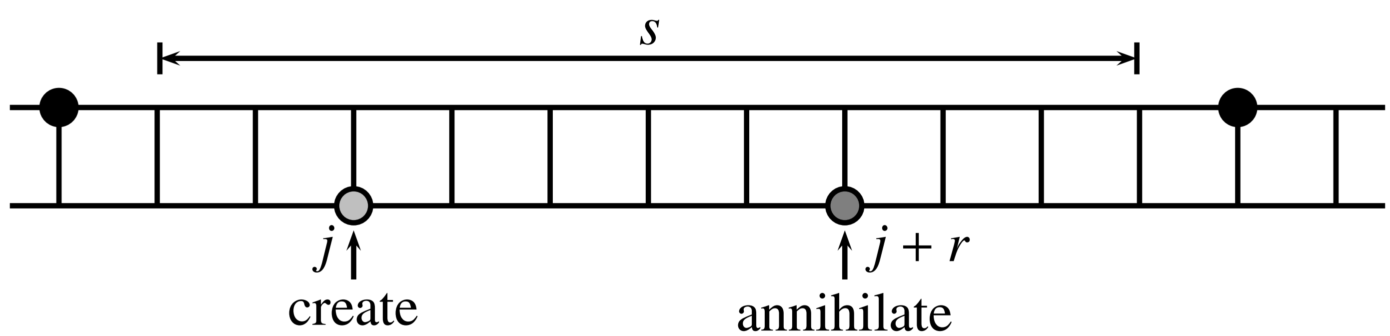
SC– Correlations



FL Correlations

- To contribute to $\langle c_{i,j}^{\dagger} c_{i,j+r} \rangle$, there must be no spinless fermions (on either legs) between rung j, where spinless fermion will be created, and rung $j+r$, where spinless fermion will be annihilated.
- Configurations satisfying this condition are those in which rung $j+r$ sits in a gap of length $s \geq r$.





FL Correlations

- Can write FL correlation as

$$\langle c^{\dagger}_{i,j} c_{i,j+r} \rangle = \sum_{s} P(s) \sum_{s'_i, s'_f} \psi^*(s'_f)\psi(s'_i),$$

where $P(s)$ is probability of finding a gap of length s in ground state, and $\psi(s')$ is 'amplitude' of single spinless fermion at site s' within gap.

- $\sum_{s'_i, s'_f} \psi^*(s'_f)\psi(s'_f)$ is $O(1)$ number, so $\langle c^{\dagger}_{i,j} c_{i,j+r} \rangle \sim \sum_s P(s)$.

- Gap of s rungs on ladder $\mapsto$ gap of s sites on chain.

- From SAC and C. L. Henley, Phys. Rev. B **69**, 075112 (2004), know that probability of a gap of s sites in 1D Fermi sea is

$$P(s) = \det(\mathbb{1} - G_C(s)),$$

where $G_C(s)$ is cluster Green-function matrix.

FL Correlations

- For $s \gg 1$, know that

$$P(s) \approx \exp\left\{-s \int_0^{\bar{n}_1} f(1-\bar{n}_1, x)\, dx\right\},$$

and thus

$$\langle c_{i,j}^{\dagger} c_{i,j+r}\rangle \sim \frac{\exp\left(-r \int_0^{\bar{n}_1} f(1-\bar{n}_1, x)\, dx\right)}{1-\exp\left(-\int_0^{\bar{n}_1} f(1-\bar{n}_1, x)\, dx\right)},$$

i.e. FL correlation decays exponentially with separation r.

Summary of Correlation Exponents

limit	correlation function	correlation exponent	wave vector
$t_\perp \ll t_\parallel, t' = 0$	CDW+	2	0
		2	$2k_F$
	CDW−	$\frac{1}{2}$	$2k_F$
		2	0
	SC+	2	0
		2	$2k_F$
	SC−	$\frac{5}{2}$	$2k_F$
		4	0

Strong Inter-leg Hopping Limit

- When $t_\perp \gg t_\parallel$, $t' = 0$, spinless fermions very nearly localized onto rungs of ladder, hopping to adjacent rungs only very rarely.
- Each spinless fermion very nearly in rung ground state

$$|+, j\rangle = \frac{1}{\sqrt{2}}\left(c^\dagger_{1,j} + c^\dagger_{2,j}\right)|0\rangle = C^\dagger_j\,|0\rangle\,.$$

- Call spinless fermion in rung ground state rung fermion.
- Essentially problem of 1D rung fermions with infinite nearest-neighbor repulsion.
- Use trio of maps to write rung-fermion ground state

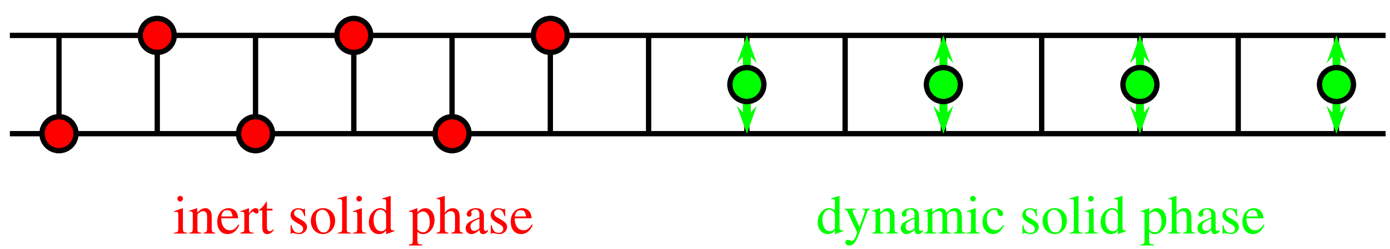
$$|\Psi\rangle = \sum\nolimits_{j_1<\cdots<j_P} \Psi_F(k_1,\ldots,k_P; j_1,\ldots,j_P)\, C^\dagger_{j_1} C^\dagger_{j_2+1} \cdots C^\dagger_{j_P+P-1}\,|0\rangle$$

in terms of 1D Fermi sea.

Long-Range Order and Phase Separation

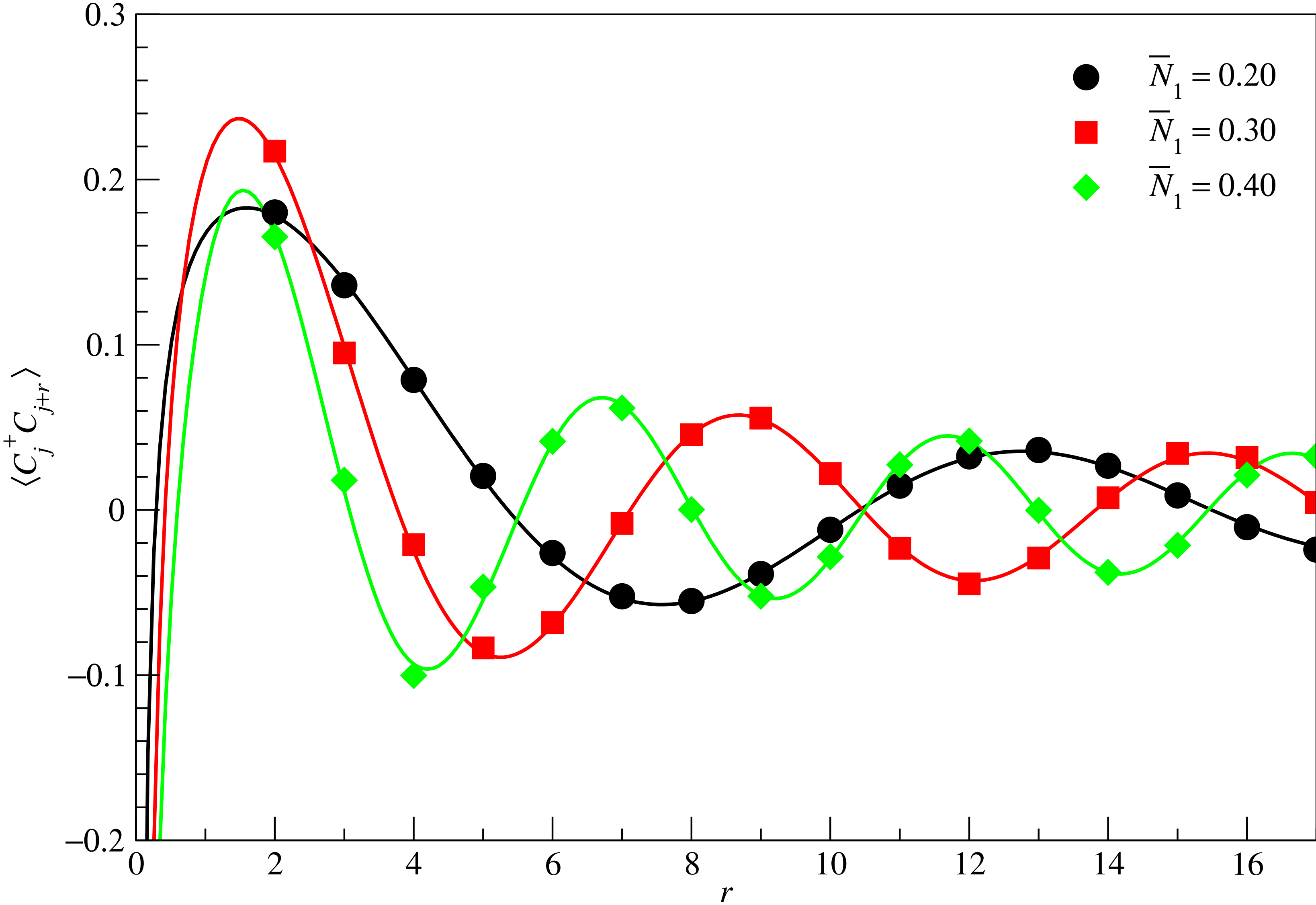
- At $\bar{n}_2 = \frac{1}{4} \equiv \bar{n}_1 = \frac{1}{2}$, every other rung occupied. Spinless fermions can continue to hop back and forth along rung, but cannot hop to adjacent rungs (infinite nearest-neighbor repulsion). Dynamic solid phase with long-range CDW order.

- For $\bar{n}_2 > \frac{1}{4}$, a fraction of spinless fermions become immobile (inert solid phase, $\bar{n}_2 = \frac{1}{2} \equiv \bar{n}_1 = 1$), while the rest remain in dynamic solid phase, $\bar{n}_2 = \frac{1}{4}$.

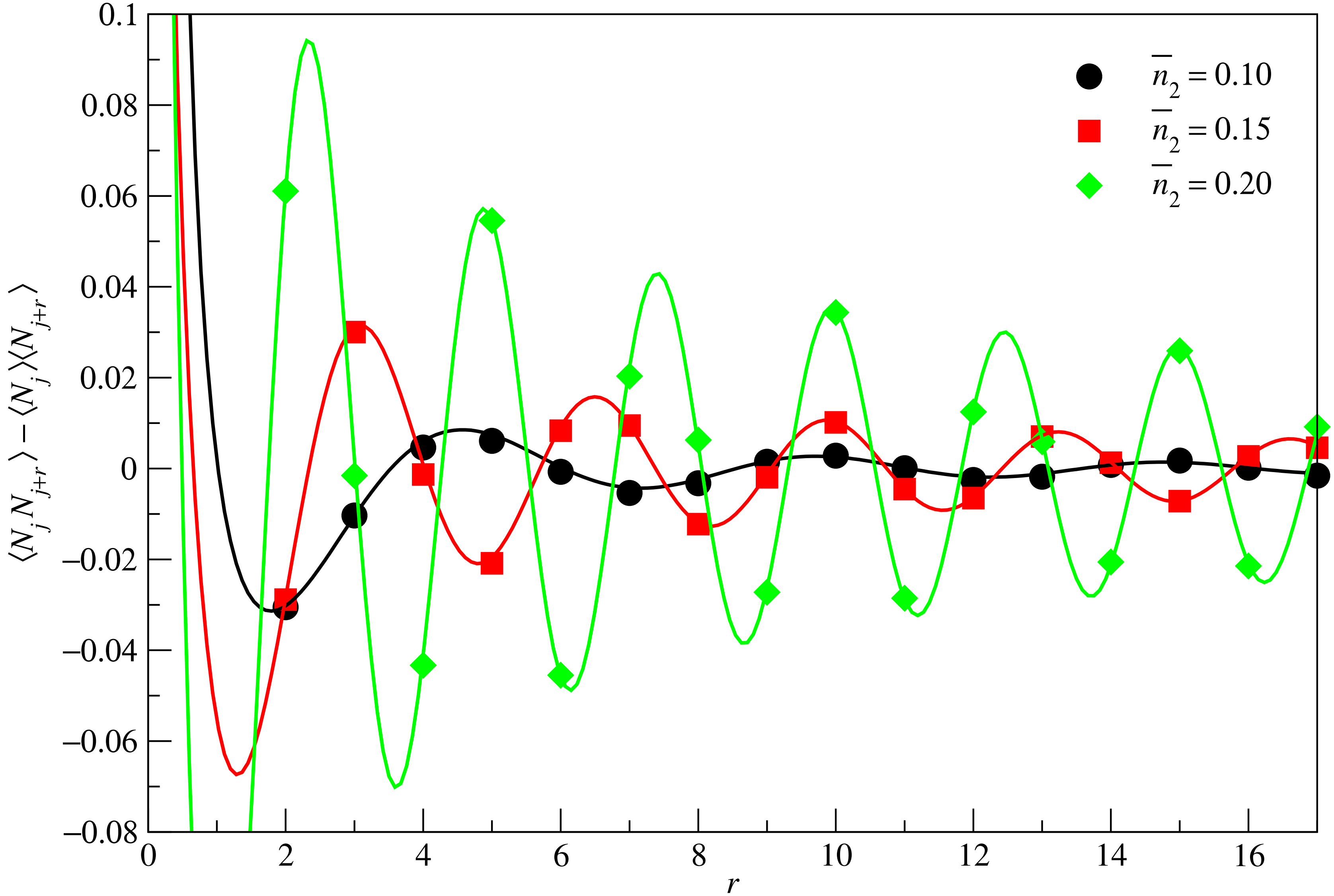


- For given $\bar{n}_2$, ground-state composition of dynamic and inert solid phases determined by having as many spinless fermions in dynamic solid phase as possible.

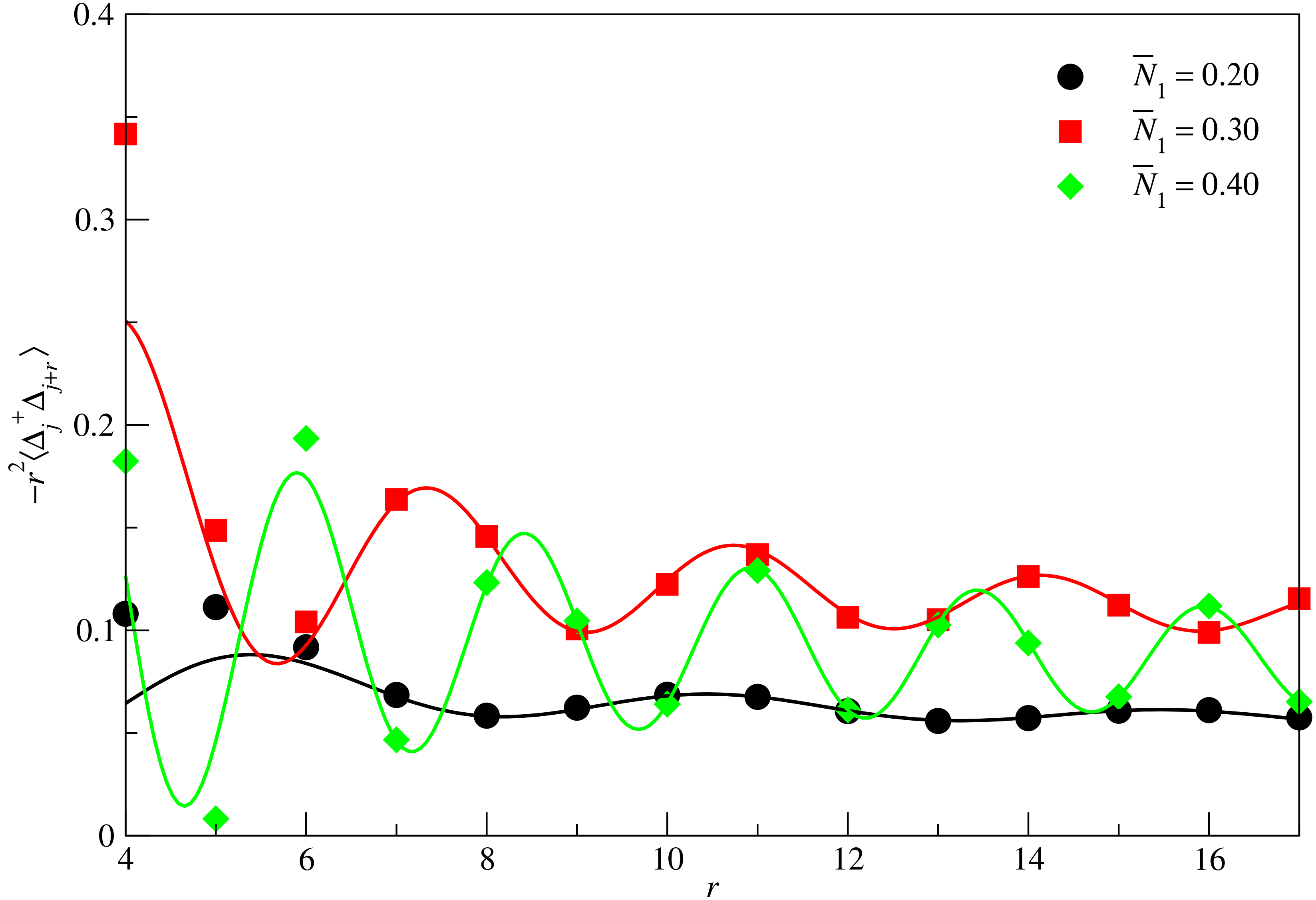
FL Correlations



CDW Correlations



SC Correlations



Summary of Correlation Exponents

limit	correlation function	correlation exponent	wave vector
$t_\perp \gg t_\parallel, t' = 0$	FL	$\frac{1}{4}$	k_F
		1	k_F
	CDW	$\frac{1}{2}$	$2k_F$
		2	0
		2	$2k_F$
	SC	$\frac{1}{8}$	0
		$\frac{1}{4}$	$2k_F$
		2	0
		2	$2k_F$

Conclusions

- Exact solution via

  (i) right-exclusion configuration-to-configuration map;
  (ii) Bloch-state-to-Bloch-state map; and
  (iii) wave-vector-to-wave-vector map

  relating nearest-neighbor excluded chain and nearest-neighbor included chain.

- Corresponding observables and intervening-particle expansion allows some correlation functions to be calculated, either analytically or numerically.

- Study three limiting cases of the extended Hubbard ladder of spinless fermions:

  (i) strong correlated hopping;
  (ii) weak inter-leg hopping; and
  (iii) strong inter-leg hopping.

Conclusions

- Wrote down exact ground states, calculated various correlation functions, and perform nonlinear curve fitting to get correlation exponents.
- Many unexpected universal correlation exponents not found in existing literature on Luttinger liquids.
- Hard-core boson two-point function maps to nonlocal string observable in 1D Fermi sea. Correlation exponent $\beta = \frac{1}{2}$ calculated by Efetov and Larkin an example of string correlation exponent.
- Numerical results hints at rich physics of nonlocal string observables.