Exact Ground States and Correlation Functions of Chain and Ladder Models of Interacting Hardcore Bosons and Spinless Fermions

CHEONG Siew Ann[1] & Christopher L. HENLEY[2]

[1]Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University;

[2]Laboratory of Atomic and Solid State Physics, Cornell University

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NNI and NNE Chain Models

- Nearest-neighbor included (NNI) chain

$$H_b = -t\sum_j \left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j\right) + U\sum_j n_j(n_j - \mathbb{1}), \quad U \to \infty;$$
$$H_c = -t\sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right),$$

  of hardcore bosons (H_b) or spinless fermions (H_c).

- Ground state of H_b solved exactly in terms of 1D Fermi sea ground state of H_c using Jordan-Wigner transformation [P. Jordan and E. Wigner, *Z. Phys.* **47**, 631 (1928)].

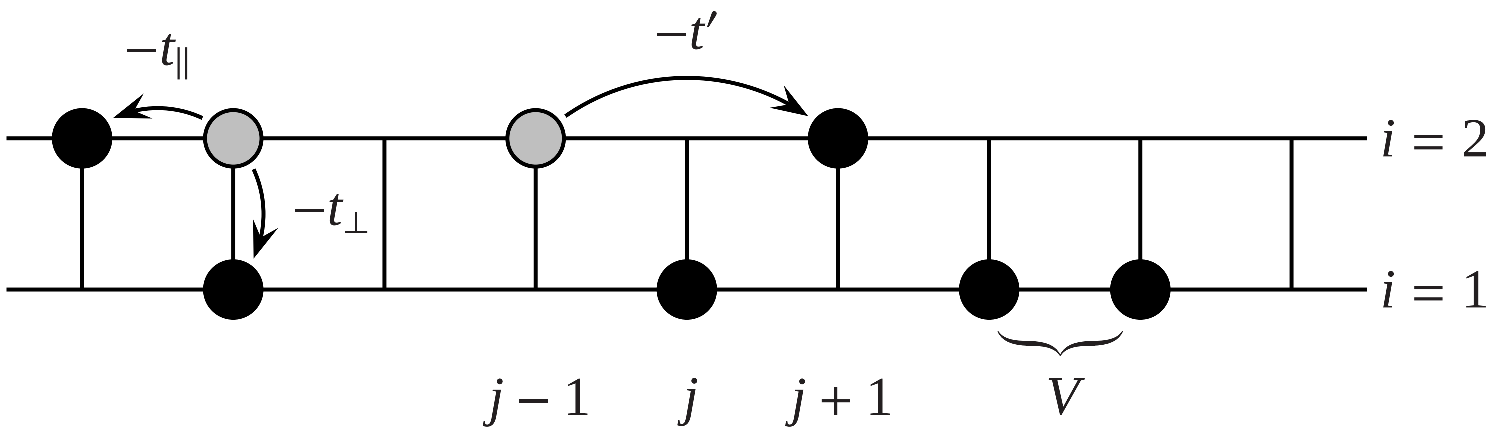
- Nearest-neighbor excluded (NNE) chain

$$H_A = H_a + V\sum_j n_j n_{j+1}, \quad V \to \infty,$$

  of hardcore bosons ($A = B$, $a = b$) or spinless fermions ($A = C$, $a = c$).

- $L \to \infty$ sites with $P \to \infty$ particles with $\bar{N} = P/L$ fixed. Closed shell boundary conditions: identical treatment for bosons and fermions.

Ladder Model of Spinless Fermions

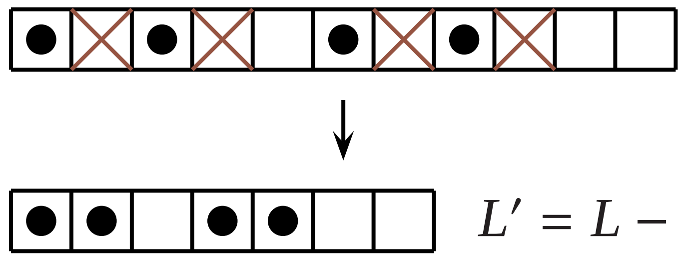


$$
\begin{aligned}
H_{t_{\|}t_{\perp}t'V} = {} & -t_{\|}\sum_i\sum_j\left(c^\dagger_{i,j}c_{i,j+1} + c^\dagger_{i,j+1}c_{i,j}\right) - t_{\perp}\sum_i\sum_j\left(c^\dagger_{i,j}c_{i+1,j} + c^\dagger_{i+1,j}c_{i,j}\right) \\
& - t'\sum_i\sum_j\left(c^\dagger_{i,j}n_{i+1,j+1}c_{i,j+2} + c^\dagger_{i,j+2}n_{i+1,j+1}c_{i,j}\right) \\
& - t'\sum_i\sum_j\left(c^\dagger_{i+1,j}n_{i,j+1}c_{i+1,j+2} + c^\dagger_{i+1,j+2}n_{i,j+1}c_{i+1,j}\right) \\
& + V\sum_i\sum_j n_{i,j}n_{i,j+1} + V\sum_i\sum_j n_{i,j}n_{i+1,j}, \qquad V \to \infty.
\end{aligned}
$$

Constructive Sequence of Maps

- Right-exclusion map: $\langle\alpha|H_A|\beta\rangle = \langle\alpha'|H_a|\beta'\rangle$ if $|\alpha\rangle \mapsto |\alpha'\rangle$ and $|\beta\rangle \mapsto |\beta'\rangle$, but not one-to-one. [P. Fendley, B. Nienhuis, and K. Schoutens, *J. Phys. A: Math. Gen* **36**, 12399 (2003)]

NNE $|\alpha\rangle$ $L = 11, P = 4$



NNI $|\alpha'\rangle$ $L' = L - P = 7, P' = P = 4$

- Bloch-state-to-Bloch-state map: for $|\alpha\rangle \mapsto |\alpha'\rangle$,

$$|\alpha; q\rangle = \frac{1}{\sqrt{\mathcal{N}_\alpha(q)}} \sum_{j=1}^{L} e^{-\mathrm{i}\,q\,j}\, T_j\,|\alpha\rangle \mapsto |\alpha'; q'\rangle = \frac{1}{\sqrt{\mathcal{N}'_{\alpha'}(q')}} \sum_{j'=1}^{L'} e^{-\mathrm{i}\,q'\,j'}\, T_{j'}\,|\alpha'\rangle\,,$$

where T_j is translation operator.

- Wave-vector-to-wave-vector map: $\langle\alpha; q|H_A|\beta; q\rangle = \langle\alpha'; q'|H_a|\beta'; q'\rangle$ for all $|\alpha\rangle \mapsto |\alpha\rangle$ and $|\beta\rangle \mapsto |\beta'\rangle$ if

$$q = \frac{2\pi n}{L} \mapsto q' = \frac{2\pi n}{L'}, \quad n \in \mathbb{Z}.$$

The Intervening-Particle Expansion

- Exact solution of NNE chain (density $\bar{N}$) in terms of NNI chain (density $\bar{n}$). Identical energy spectra for H_B, H_C, H_b, H_c.
- NNE expectation $\langle O \rangle$ and NNI expectation $\langle O' \rangle$ related by $\langle O \rangle / \bar{N} = \langle O' \rangle / \bar{n}$ if $\sqrt{l_\alpha l_\beta}\langle \alpha; q | O | \beta; q \rangle = \sqrt{l'_{\alpha'} l'_{\beta'}}\langle \alpha'; q' | O' | \beta'; q' \rangle$, where l_α and $l'_{\alpha'}$ are periods of $|\alpha\rangle$ and $|\alpha'\rangle$ respectively.
- For products of NNE chain local operators, write

$$\langle O_j O_{j+r} \rangle = \sum_{\{p\}} \langle O_j O_p O_{j+r} \rangle = \frac{\bar{N}}{\bar{n}} \sum_{\{p\}} \langle O'_j O'_p O'_{j+r-p} \rangle ,$$

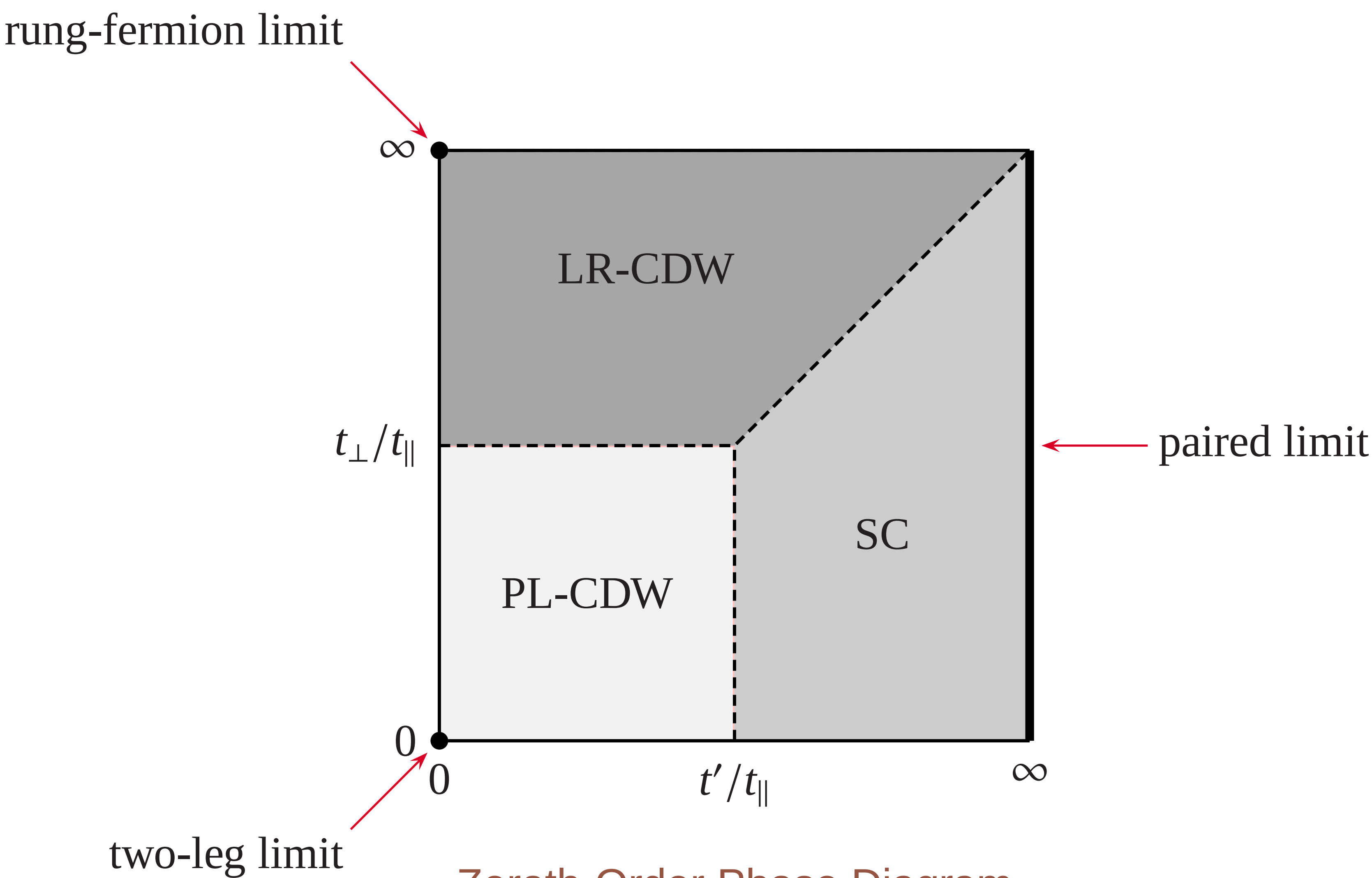
  where O_p is product of p particle number operators N_i and $(r-p)$ hole number operators $(\mathbb{1} - N_i)$, and sum is over all possible intervening particle configurations between j and $j+r$.
- Simple rules for mapping NNE conditional expectation $\langle O_j O_p O_{j+r} \rangle$ to NNI condition expectation $\langle O'_j O'_p O'_{j+r-p} \rangle$.
- Intervening-particle expansion summed numerically, then nonlinear curve fitting to expected asymptotic forms.

Correlations in Bosonic/Fermionic NNE Chains

	hardcore boson		spinless fermion		
correlation	exponent	wave vector	exponent	wave vector	correlation
$\langle B_j^\dagger B_{j+r}\rangle$	$\frac{1}{2}$	0	$1 \to \frac{1}{4}$	k_F	$\langle C_j^\dagger C_{j+r}\rangle$
$\langle N_j N_{j+r}\rangle$	$\frac{1}{2} + \frac{5}{2}(\frac{1}{2} - \overline{N})$	$2k_F$	$\frac{1}{2} + \frac{5}{2}(\frac{1}{2} - \overline{N})$	$2k_F$	$\langle N_j N_{j+r}\rangle$
$\langle \Delta_j^\dagger \Delta_{j+r}\rangle$	$\frac{7}{4}$	$2k_F$	$\frac{7}{4}$	$2k_F$	$\langle \Delta_j^\dagger \Delta_{j+r}\rangle$

- $N_j = B_j^\dagger B_j$ and $\Delta_j = B_j^\dagger B_{j+2}^\dagger$ for hardcore boson, whereas $N_j = C_j^\dagger C_j$ and $\Delta_j = C_j^\dagger C_{j+2}^\dagger$ for spinless fermions.
- Only exponents and wave vectors of leading asymptotic behaviours shown. Mixture of universal and nonuniversal exponents.
- $k_F = \pi\overline{N}$ consequence of Luttinger's theorem.
- $\langle B_j^\dagger B_{j+r}\rangle \sim r^{-\frac{1}{2}}$ previously obtained by Efetov and Larkin. [K. B. Efetov and A. I. Larkin, *Sov. Phys. JETP* **42**, 390 (1976)].

Limiting Cases for the Ladder Model



Zeroth-Order Phase Diagram

Ladder Correlations in the Three Limiting Cases

limit	correlation	exponent	wave vector
paired, $t' \gg t_{\|\|}, t_\perp$	FL	∞	-
	CDW-π	$\frac{1}{2} + \frac{5}{2}\left(\frac{1}{2} - \bar{N}_1\right)$	$2k_F$
	SC	$\frac{1}{2}$	0
two-leg, $t_\perp \ll t_{\|\|}, t' = 0$	FL	∞	-
	CDW+ CDW−	2 $\frac{1}{2}$	0, $2k_F$ $2k_F$
	SC+ SC−	2 $\frac{5}{2}$	0, $2k_F$ $2k_F$
rung-fermion, $t_\perp \gg t_{\|\|}, t' = 0$	FL	$1 \to \frac{1}{4}$	k_F
	CDW	$\frac{1}{2} + \frac{5}{2}(\frac{1}{2} - \bar{N})$	$2k_F$
	SC	$\frac{7}{4}$	$2k_F$

Conclusions

- Exact solutions of NNE chain in terms of NNI chain, for hardcore bosons and spinless fermions, and ultimately in terms of 1D Fermi sea, via constructive sequence of maps.
- Exact solutions of ladder model of spinless fermions in three limiting cases, by mapping onto NNE chains, or directly onto 1D Fermi sea.
- Intervening-particle expansion of NNE chain expectations in terms of NNE chain expectations, for calculating NNE chain correlations.
- Nonlinear curve fitting gives universal and nonuniversal exponents not expected from Luttinger liquid paradigm.
- Some point correlations on NNE chain map to point correlations, others map to string correlations in 1D Fermi sea.
- Conjecture that in all exact solutions (chain + ladder), all exponents universal, and are rational polynomials of universal Fermi-liquid parameter $K = 1$.