9. Contour Integration

Contour integration is a powerful technique, based on complex analysis, that allows us to solve certain integrals that are otherwise hard or impossible to solve. Contour integrals also have important applications in physics, particularly in the study of waves and oscillations.

9.1 Contour integrals

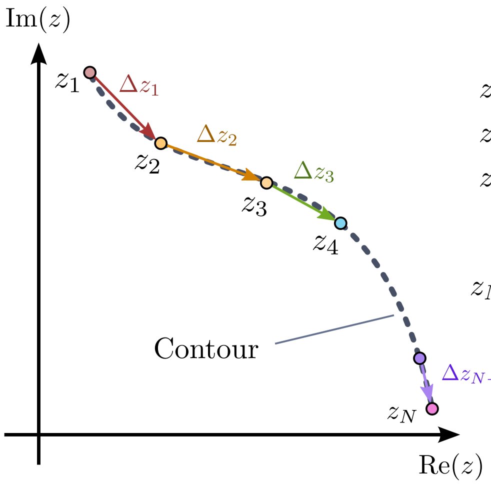
Recall that for a real function $f(x)$, the definite integral from $x = a$ to $x = b$ is the area under the curve between those two points. As discussed in Chapter 3, the integral can be expressed as a limit expression: we divide the interval into N segments of width Δx, and find the sum of $\Delta x\, f(x_n)$ in the $N \to \infty$ limit:

$$\int_a^b dx\; f(x) \;=\; \lim_{N\rightarrow 0} \sum_{n=0}^{N} \Delta x\; f(x_n), \quad \text{where} \quad x_n = a + n\Delta x, \;\; \Delta x = \frac{b-a}{N}. \tag{9.1}$$

Now suppose f is a complex function of a complex variable. A straightfoward way to define the integral of $f(z)$ is to adopt an analogous expression:

$$\lim_{N\rightarrow 0} \sum_{n=0}^{N} \Delta z\; f(z_n)$$

But since f takes complex inputs, the z_n's don't have to be on the real line. So instead of going from a to b along the real line, we can imagine chaining together a sequence of points $z_1, z_2, \dots, z_N$ in the complex plane, separated by displacements $\Delta z_1, \Delta z_2, \Delta z_3, \dots, \Delta z_{N-1}$:



$$\begin{aligned} z_2 &= z_1 + \Delta z_1, \\ z_3 &= z_2 + \Delta z_2, \\ z_4 &= z_3 + \Delta z_3, \\ &\;\vdots \\ z_N &= z_{N-1} + \Delta z_{N-1}. \end{aligned}$$

Then the sum we are interested in is

$$\sum_{n=1}^{N-1} \Delta z_n\; f(z_n) \;=\; \Delta z_1\, f(z_1) + \Delta z_2\, f(z_2) + \cdots + \Delta z_{N-1}\, f(z_{N-1}). \tag{9.2}$$

In the limit $N \to \infty$, each displacement Δz_n becomes infinitesimal, and the sequence of points $\{z_1, z_2, \dots, z_N\}$ becomes a continuous trajectory in the complex plane (see Section 3.6). Such a trajectory is called a **contour**.

Let us denote a given contour by an abstract symbol, such as Γ. Then the **contour integral** over Γ is defined as

$$\int_\Gamma f(z)\, dz \equiv \lim_{N\to\infty} \sum_{n=1}^{N-1} \Delta z_n \; f(z_n). \tag{9.3}$$

The symbol Γ in the subscript of the integral sign indicates that the integral takes place over the contour Γ. When defining a contour integral, it is always necessary to specify which contour we are integrating over. Whereas for definite real integrals we only needed to specify the end-points, for a contour integral it is not enough to just state the end-points—we need to specify the entire contour that starts and ends at those points!

Moreover, the specification of a contour must include the direction along which the curve is traversed. If we integrate along the same curve in the opposite direction, the value of the contour integral switches sign (this is similar to swapping the end-points of a definite real integral).

Note—Unlike a definite real integral, a contour integral does not have a geometrical interpretation in terms of an "area under a curve". It should be thought of as a generalization of the algebraic (as opposed to geometric) concept of an integral.

Note—The contour should not be mistaken for the graph of the integrand! In many contour integration problems, the contour is sketched for you as a curve in the complex plane. Students who are new to contour integration sometimes mistake this for the graph of the integrand.

9.1.1 Contour integral along a parametric curve

Simple contour integrals can be calculated by **parameterizing the contour**. Consider a contour integral

$$\int_\Gamma dz\; f(z),$$

where f is a complex function of a complex variable and Γ is a given contour.

As discussed in Section 3.6, a trajectory in the complex plane can be described by a complex function of a *real* variable, $z(t)$:

$$\Gamma \equiv \Big\{ z(t) \;\Big|\; t_1 < t < t_2 \Big\}, \quad \text{where } \; t \in \mathbb{R},\, z(t) \in \mathbb{C}. \tag{9.4}$$

The real numbers t_1 and t_2 specify two complex numbers, $z(t_1)$ and $z(t_2)$, which are the end-points of the contour. The rest of the contour consists of the values of $z(t)$ between those end-points. Provided we can parameterize Γ in such a manner, the complex displacement dz in the contour integral can be written as

$$dz \to dt\, \frac{dz}{dt}. \tag{9.5}$$

Then we can express the contour integral over Γ as a definite integral over t:

$$\int_\Gamma dz\; f(z) = \int_{t_1}^{t_2} dt\; \frac{dz}{dt}\, f\big(z(t)\big). \tag{9.6}$$

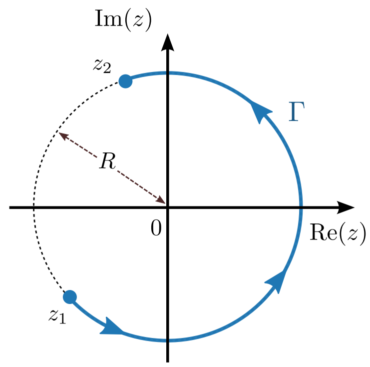
This can then be calculated using standard integration techniques such as those discussed in Chapter 3. A simple example is given in the next section.

9.1.2 A contour integral over a circular arc

Let us use the method of parameterizing the contour to calculate the contour integral

$$\int_{\Gamma[R,\theta_1,\theta_2]} dz\; z^n, \;\; n \in \mathbb{Z}, \tag{9.7}$$

where the contour $\Gamma[R,\theta_1,\theta_2]$ is a counter-clockwise arc of radius $R > 0$, from $z_1 = Re^{i\theta_1}$ to $z_2 = Re^{i\theta_2}$:



We can parameterize the contour as follows:

$$\Gamma[R,\theta_1,\theta_2] = \big\{z(\theta) \;\big|\; \theta_1 \le \theta \le \theta_2\big\}, \quad \text{where } z(\theta) = Re^{i\theta}. \tag{9.8}$$

Then the contour integral can be converted into a definite integral over the real variable θ:

$$\int_{\Gamma[R,\theta_1,\theta_2]} dz\; z^n = \int_{\theta_1}^{\theta_2} d\theta\; z^n \,\frac{dz}{d\theta} \tag{9.9}$$

$$= \int_{\theta_1}^{\theta_2} d\theta\; \left(Re^{i\theta}\right)^n \left(iR\,e^{i\theta}\right) \tag{9.10}$$

$$= iR^{n+1} \int_{\theta_1}^{\theta_2} d\theta\; e^{i(n+1)\theta}. \tag{9.11}$$

To proceed, there are two cases that we must treat separately. First, for $n \ne -1$,

$$\int_{\theta_1}^{\theta_2} d\theta\; e^{i(n+1)\theta} = \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_{\theta_1}^{\theta_2} = \frac{e^{i(n+1)\theta_2} - e^{i(n+1)\theta_1}}{i(n+1)}. \tag{9.12}$$

Second, we have the case $n = -1$. This cannot be handled by the above equations, since the factor of $n+1$ in the denominator would vanish. Instead,

$$\int_{\theta_1}^{\theta_2} d\theta\; \Big[e^{i(n+1)\theta}\Big]_{n=-1} = \int_{\theta_1}^{\theta_2} d\theta \;=\; \theta_2 - \theta_1. \tag{9.13}$$

Putting the two cases together, we arrive at the result

$$\int_{\Gamma[\theta_1,\theta_2]} dz\; z^n = \begin{cases} i(\theta_2-\theta_1), & \text{if } n = -1 \\ R^{n+1}\,\dfrac{e^{i(n+1)\theta_2} - e^{i(n+1)\theta_1}}{n+1}, & \text{if } n \ne -1. \end{cases} \tag{9.14}$$

The case where $\theta_2 = \theta_1 + 2\pi$ is of particular interest. Here, Γ forms a complete loop, and the result simplifies to

$$\oint_\Gamma dz\; z^n = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{if } n \ne -1, \end{cases} \tag{9.15}$$

which is independent of R as well as the choice of θ_1 and θ_2. (Here, the special integration symbol $\oint$ is used to indicate that the contour integral is taken over a loop.) This is a very important result that we will make ample use of later.

By the way, what if n is not an integer? In that case, the integrand z^n is multi-valued (as we saw in Chapter 8). This is problematic, since the definition of a contour integral assumes the integrand is a well-defined function. To get around this, we can specify a branch cut and perform the contour integral with any of the branch functions of z^n. So long as the branch cut avoids intersecting with the contour Γ, the result obtained above by parametrizing the contour integral remain valid. However, Γ cannot properly be taken along a complete loop, as that would entail crossing the branch cut.

9.2 Cauchy's integral theorem

A **loop integral** is a contour integral taken over a loop in the complex plane; i.e., with the same starting and ending point. We have already seen the case of a circular loop integral in the previous example.

Loop integrals play an important role in complex analysis. This importance stems from the following property, known as **Cauchy's integral theorem**:

> If $f(z)$ is analytic everywhere inside a loop Γ, then $\displaystyle\oint_\Gamma dz\; f(z) = 0$.

9.2.1 Proof of Cauchy's integral theorem

Cauchy's integral theorem can be derived from Stokes' theorem, which states that for any differentiable vector field $\vec{A}(x, y, z)$ defined within a three-dimensional space, its line integral around a loop Γ is equal to the flux of its curl through any surface enclosed by the loop. Mathematically, this is stated as

$$\oint_\Gamma d\vec{\ell} \cdot \vec{A} = \int_{S(\Gamma)} d^2r\; \hat{n} \cdot \left(\nabla \times \vec{A}\right), \tag{9.16}$$

where $S(\Gamma)$ denotes a two-dimensional surface enclosed by the loop Γ, and $\hat{n}$ denotes a normal vector sticking out of the surface at each integration point.

We only need the 2D version of Stokes' theorem, in which both the loop Γ and the enclosed surface $S(\Gamma)$ are restricted to the $x-y$ plane, and $\vec{A}(x, y)$ likewise has no z component. Then Stokes' theorem simplifies to

$$\oint_\Gamma d\vec{\ell} \cdot \vec{A} = \iint_{S(\Gamma)} dx\, dy\, \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \tag{9.17}$$

Now consider a loop integral

$$\oint_\Gamma dz\; f(z),$$

where Γ is a loop trajectory and $f(z)$ is some analytic function that is analytic in the two-dimensional domain $S(\Gamma)$ enclosed by Γ. Let us decompose f into its real and imaginary parts,

$$f(x + iy) = u(x, y) + iv(x, y). \tag{9.18}$$

The analyticity of f implies that the real functions u and v are differentiable and obey the Cauchy-Riemann equations (see Section 6.3).

Let us manipulate the loop integral as follows:

$$\oint_\Gamma dz\, f(z) = \oint_\Gamma (dx + idy)\;(u + iv) \tag{9.19}$$

$$= \oint_\Gamma \Big(dx\,(u+iv) + dy\,(iu - v)\Big) \tag{9.20}$$

$$= \oint_\Gamma \begin{bmatrix}dx\\dy\end{bmatrix} \cdot \begin{bmatrix}u+iv\\iu-v\end{bmatrix} \tag{9.21}$$

$$= \oint_\Gamma d\vec{\ell}\cdot\vec{A}. \tag{9.22}$$

The last expression is a line integral involving the complex vector field

$$\vec{A}(x,y) = \begin{bmatrix}u(x,y)+iv(x,y)\\iu(x,y)-v(x,y)\end{bmatrix}. \tag{9.23}$$

Using Stokes' theorem in 2D, we convert this into the area integral

$$\iint_{S(\Gamma)} dx\,dy\, \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \iint_{S(\Gamma)} dx\,dy\, \left[\left(i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] \tag{9.24}$$

$$= \iint_{S(\Gamma)} dx\,dy\, \left[-\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right]. \tag{9.25}$$

On the last line, the two terms in parenthesis are both zero because, according to the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\;\text{and}\;\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{9.26}$$

Hence, the loop integral is zero. Q.E.D.

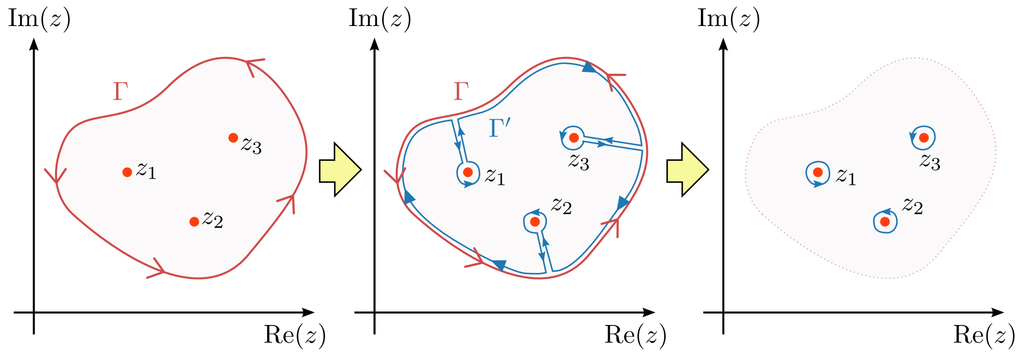
9.2.2 Consequences of Cauchy's integral theorem

Cauchy's integral theorem does not apply if the integrand $f(z)$ is non-analytic somewhere inside the loop. In particular, suppose $f(z)$ is not differentiable at one or more discrete points inside the loop, $\{z_1, z_2, \dots\}$. Then we can show that

$$\oint_\Gamma dz\; f(z) \;=\; \sum_n \oint_{z_n} dz\; f(z), \tag{9.27}$$

where $\oint_{z_n}$ denotes an integral over a loop of infinitesimal radius around the n-th point of non-analyticity, in the *same direction* (i.e., clockwise or counter-clockwise) as Γ.

The proof is based on the figure below. The red loop, Γ, is the contour we want to integrate over. The integrand is analytic throughout the enclosed area except at several discrete points, say $\{z_1, z_2, z_3\}$. Let us define a new loop contour, Γ', shown by the blue loop. It follows the same curve as Γ but with the following differences: (i) it circles in the *opposite* direction from Γ, (ii) it contains tendrils that extend from the outer curve to each point of non-analyticity, and (iii) each tendril is attached to an infinitesimal loop encircling a point of non-analyticity.



The loop Γ' encloses no points of non-analyticity, so Cauchy's integral theorem says that the integral over it is zero. But the contour integral over Γ' can be broken up into three pieces: (i) the part that follows Γ but in the opposite direction, (ii) the tendrils, and (iii) the infinitesimal inner loops:

$$\oint_{\Gamma'} dz\, f(z) \;=\; 0 \quad \text{(by Cauchy's Integral Theorem)} \tag{9.28}$$

$$= \int_{\text{big loop}} dz\, f(z) + \int_{\text{tendrils}} dz\, f(z) + \sum_{\text{small loop } n} \oint_{z_n} dz\, f(z) \tag{9.29}$$

The first term is equal to the negative of $\oint_\Gamma dz\, f(z)$, since it follows a contour that is just like Γ except going the other way. The second term is zero, because each tendril consists of two contours taken in opposite directions, which cancel. Thus, the above equation reduces to

$$\oint_\Gamma dz\, f(z) = \sum_n \oint_{z_n} dz\, f(z). \tag{9.30}$$

The loop contour integral over Γ is equal to the sum of infinitesimal loop contour integrals encircling each point of non-analyticity. Notably, each of the infinitesimal loops circles in the *same* direction as Γ (e.g., counter-clockwise in the above figure).

Another way of thinking about this is that Cauchy's integral theorem says regions of analyticity don't count towards the value of a loop integral. Hence, we can contract a loop across any domain in which $f(z)$ is analytic, until the contour becomes as small as possible. This contraction replaces Γ with a discrete set of infinitesimal loops enclosing the points of non-analyticity.

9.3 Poles

In the previous section, we referred to situations where $f(z)$ is non-analytic at discrete points. "Discrete", in this context, means that each point of non-analyticity is surrounded by a finite region over which $f(z)$ is analytic, isolating it from other points of non-analyticity. Such situations commonly arise from functions like

$$f(z) = \frac{1}{(z-z_0)^n}, \quad \text{where } n \in \{1, 2, 3, \dots\}. \tag{9.31}$$

For $z = z_0$, this function is non-analytic because its value is singular. We say that there is a **pole** at z_0. The integer n is called the **order** of the pole.

9.3.1 Residue of a simple pole

Poles of order 1 are called **simple poles**, and they are of special interest. If a function has a simple pole at z_0, it can be approximated near the pole as

$$f(z) \approx \frac{A}{z - z_0}. \tag{9.32}$$

The complex numerator A is called the **residue** of the pole (so-called because it's what's left-over if we take away the singular factor corresponding to the pole.) We denote the residue of a function $f(z)$ at $z = z_0$ by

$$\mathrm{Res}\big[\, f(z),\, z = z_0 \,\big]. \tag{9.33}$$

Note that if f is analytic at z_0 (i.e., there is no pole there), then $\mathrm{Res}[f(z), z = z_0] = 0$.

Example—Consider the function

$$f(z) = \frac{5}{i - 3z}. \tag{9.34}$$

To find the pole and residue, divide the numerator and denominator by -3:

$$f(z) = \frac{-5/3}{z - i/3}. \tag{9.35}$$

Thus, there is a simple pole at $z = i/3$ and

$$\mathrm{Res}\big[\, f(z),\, z = i/3 \,\big] = -5/3. \tag{9.36}$$

Example—Consider the function

$$f(z) = \frac{z}{z^2 + 1}. \tag{9.37}$$

To find the poles and residues, we factorize the denominator:

$$f(z) = \frac{z}{(z + i)(z - i)}. \tag{9.38}$$

Hence, there are two simple poles, at $z = \pm i$.

To find the residue at $z = i$, separate the divergent part from the rest of the expression:

$$f(z) = \frac{\left(\frac{z}{z+i}\right)}{z - i} \tag{9.39}$$

$$\Rightarrow \quad \mathrm{Res}\big[\, f(z), z = i \,\big] = \left[\frac{z}{z + i}\right]_{z=i} = 1/2. \tag{9.40}$$

Likewise, for the other pole at $z = -i$,

$$\mathrm{Res}\big[\, f(z),\, z = -i \,\big] = \left[\frac{z}{z - i}\right]_{z=-i} = 1/2. \tag{9.41}$$

9.3.2 The residue theorem

In Section 9.1.2, we used contour parameterization to calculate

$$\oint_\Gamma \frac{dz}{z} = 2\pi i, \tag{9.42}$$

where Γ is a counter-clockwise circular loop centered on the origin. This holds for any (non-zero) loop radius. Combining this result with Eq. (9.27), we obtain the **residue theorem**:

For any analytic function $f(z)$ with a simple pole at z_0,

$$\oint_{\Gamma[z_0]} dz\; f(z) = \pm 2\pi i\, \mathrm{Res}\big[\, f(z)\,, z = z_0\,\big], \tag{9.43}$$

where $\Gamma[z_0]$ denotes an infinitesimal loop around z_0. The $+$ sign holds for a counter-clockwise loop, and the $-$ sign for a clockwise loop.

Hence, we arrive at an integration technique called the **calculus of residues**:

1. Identify the poles of $f(z)$ in the domain enclosed by Γ.
2. Check that these are all simple poles, and that $f(z)$ has no other non-analytic behaviors (e.g. branch cuts) in the enclosed region.
3. At each pole z_n, calculate the residue $\mathrm{Res}[f(z),\, z = z_n]$.
4. The value of the loop integral is

$$\oint_\Gamma dz\; f(z) = \pm 2\pi i \sum_n \mathrm{Res}\big[\, f(z),\; z = z_n\,\big]. \tag{9.44}$$

   The plus sign holds if Γ is counter-clockwise, and the minus sign if it is clockwise.

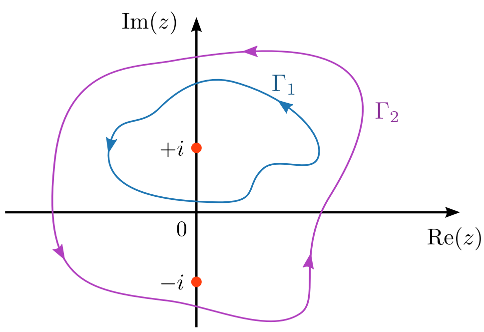
9.3.3 Example of the calculus of residues

Consider

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)\,(z-i)}. \tag{9.45}$$

By inspection, we can identify two poles: one at $+i$, with residue $1/2i$, and the other at $-i$, with residue $-1/2i$. The function is analytic everywhere else.

Suppose we integrate $f(z)$ around a counter-clockwise contour Γ_1 that encloses only the pole at $+i$, as indicated by the blue curve in the figure below:



According to the residue theorem,

$$\oint_{\Gamma_1} dz\, f(z) = 2\pi i \,\mathrm{Res}\big[\, f(z),\, z = i\,\big] \tag{9.46}$$

$$= 2\pi i \cdot \frac{1}{2i} \tag{9.47}$$

$$= \pi. \tag{9.48}$$

On the other hand, suppose we integrate around a contour Γ_2 that encloses *both* poles, as shown by the purple curve. Then the result is

$$\oint_{\Gamma_2} dz\, f(z) = 2\pi i \cdot \left[\frac{1}{2i} - \frac{1}{2i}\right] = 0. \tag{9.49}$$

9.4 Using contour integration to solve definite integrals

One of the principal applicaitons of the calculus of residues is to help us handle definite integrals *over the real domain* by converting them into contour integrals, which are then solved with the residue theorem.

As an example, consider the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}. \tag{9.50}$$

This integral is taken over real values of x, and we previously solved it using a change of variables in Section 2.4. Now let's see how to solve it using contour integration.

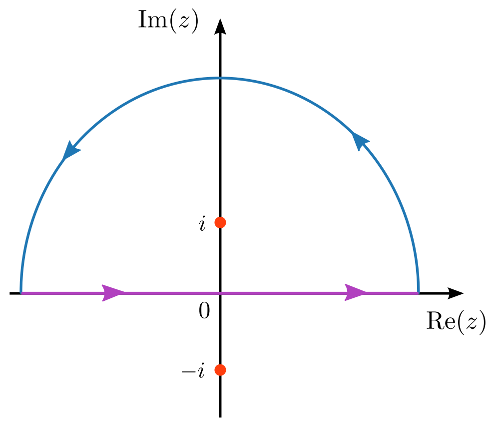
First, generalize the integrand from a function of x to an analytic function of z. (This procedure is called **analytic continuation**.) Usually, we choose the new (complex) integrand so that it reduces to the old integrand for real values of z. In this case, let

$$\frac{1}{x^2+1} \rightarrow \frac{1}{z^2+1}. \tag{9.51}$$

This is just the integrand we dealt with in the previous section.

We now have to choose the contour. The usual procedure is to define a loop contour such that one segment of the loop is the real line (from $-\infty$ to $+\infty$), and the other segment "doubles back" in the complex plane to close the loop. This is called **closing the contour**.

Here, we choose to close the contour along an anticlockwise semicircular arc in the upper half of the complex plane, as shown below:



This loop integral can be decomposed into a sum of two contour integrals:

$$\oint \frac{dz}{z^2+1} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} + \int_{\text{arc}} \frac{dz}{z^2+1}. \tag{9.52}$$

The first term is equal to the integral we're interested in. The second term is a contour integral along the big arc, and we can show that it goes to zero. To see why, observe that along an arc of radius R, the magnitude of the integrand goes as $1/R^2$, while the dz gives another factor of R (see the earlier example of parameterizing a contour integral over an arc in Section 9.1.2), so the overall integral goes as $1/R$, which vanishes as $R \to \infty$.

Now to evaluate the loop contour integral. Since the loop encloses the pole at $z = +i$,

$$\oint \frac{dz}{z^2+1} = 2\pi i \,\mathrm{Res}\left[\frac{1}{z^2+1},\, z=i\right] = \pi. \tag{9.53}$$

The loop is counterclockwise, so we take the positive sign for the residue theorem. Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi. \tag{9.54}$$

As an exercise, you can verify that closing the contour in the lower half-plane gives the same result.

9.4.1 Jordan's lemma

Before proceeding to more complicated uses of contour integration, we must introduce an important result called **Jordan's lemma**:

Let

$$I = \int_C dz\, e^{iqz}\, g(z), \tag{9.55}$$

where q is any positive real constant, and the contour C which is a semi-circular arc of radius R in the upper half-plane, centered at the origin. Then

$$\text{If } \big|g(z)\big| < g_{\max} \;\; \text{for all } \; z \in C \quad \Rightarrow \quad I \to 0 \;\; \text{as} \;\; g_{\max} \to 0. \tag{9.56}$$

In other words, if the factor of $g(z)$ in the integrand does not blow up along the arc contour (i.e., its value is bounded), then in the limit where the bounding value goes to zero, the value of the entire integral vanishes.

Usually, the limit of interest is when the radius of the arc goes to infinity. Even if the integrand vanishes in that limit, it may not be obvious that the integral I vanishes, as the integration is taken along an arc of infinite length (so we have a $0 \times \infty$ sort of situation). Jordan's lemma then proves useful, because it provides a rigorous criterion for us to conclude that I should vanish.

The proof for Jordan's lemma is tedious, and we will not go into its details.

For integrands containing a prefactor of e^{-iqz} rather than e^{iqz} (again, where $q \in \mathbb{R}^+$), a different version of Jordan's lemma holds, referring to an arc contour C' in the *lower* half-plane:

Let

$$I = \int_C dz\, e^{-iqz}\, g(z), \tag{9.57}$$

where q is any positive real constant, and the contour C which is a semi-circular arc of radius R in the lower half-plane, centered at the origin. Then

$$\text{If } \big|g(z)\big| < g_{\max} \;\; \text{for all } \; z \in C \quad \Rightarrow \quad I \to 0 \;\; \text{as} \;\; g_{\max} \to 0. \tag{9.58}$$

This is easily seen by doing the change of variable $z \to -z$ on the original form of Jordan's lemma.

As a convenient way to remember which variant of Jordan's lemma to use, think about which end of imaginary axis causes the exponential factor to vanish:

$$e^{iqz}\Big|_{z=i\infty} = e^{-\infty} = 0 \quad \Rightarrow \quad e^{iqz} \text{ vanishes far above the origin.} \tag{9.59}$$

$$e^{-iqz}\Big|_{z=-i\infty} = e^{-\infty} = 0 \quad \Rightarrow \quad e^{-iqz} \text{ vanishes far below the origin.} \tag{9.60}$$

Hence, for e^{iqz} (where q is any positive real number), the suppression occurs in the upper-half-plane. For e^{-iqz}, the suppression occurs in the lower-half-plane.

9.4.2 A contour integral using Jordan's lemma

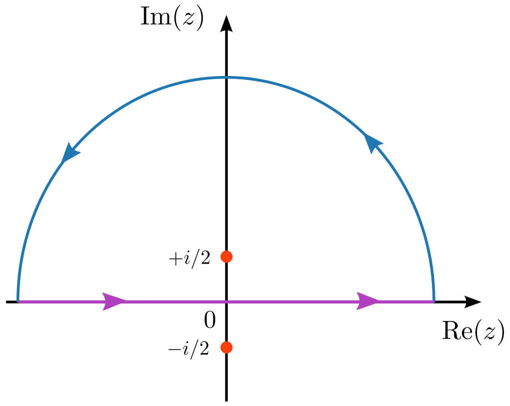
Consider the integral

$$I = \int_{-\infty}^{\infty} dx\, \frac{\cos(x)}{4x^2+1}. \tag{9.61}$$

One possible approach is to break the cosine up into $(e^{ix} + e^{-ix})/2$, and do the contour integral on each piece separately. Another approach, which saves a bit of effort, is to write

$$I = \text{Re}\left[\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{4x^2+1}\right]. \tag{9.62}$$

To do the integral, close the contour in the upper half-plane:



Then

$$\oint dz\, \frac{e^{iz}}{4z^2+1} = \int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{4x^2+1} + \int_{\text{arc}} dz\, \frac{e^{iz}}{4z^2+1}. \tag{9.63}$$

On the right-hand side, the first term is what we want. The second term is a counter-clockwise arc in the upper half-plane. According to Jordan's lemma, this term goes to zero as the arc radius goes to infinity, since the rest of the integrand goes to zero for large $|z|$:

$$\left|\frac{1}{4z^2+1}\right| \sim \frac{1}{4|z|^2} \to 0 \quad \text{as } |z| \to \infty. \tag{9.64}$$

As for the loop contour, it can be evaluated using the residue theorem:

$$\oint dz\, \frac{e^{iz}}{4z^2+1} = 2\pi i \,\text{Res}\left[\frac{1}{4}\,\frac{e^{iz}}{(z+i/2)(z-i/2)}, \; z = \frac{i}{2}\right] \tag{9.65}$$

$$= 2\pi i\, \frac{e^{-1/2}}{4i}. \tag{9.66}$$

Hence,

$$I = \text{Re}\left[\frac{\pi}{2\sqrt{e}}\right] = \frac{\pi}{2\sqrt{e}}. \tag{9.67}$$

In solving the integral this way, we must close the contour in the upper half-plane because our choice of complex integrand was bounded in the upper half-plane. Alternatively, we could have chosen to write

$$I = \text{Re}\left[\int_{-\infty}^{\infty} dx\, \frac{e^{-ix}}{4x^2+1}\right], \tag{9.68}$$

i.e., with e^{-ix} rather than e^{ix} in the numerator. In that case, Jordan's lemma tells us to close the contour in the lower half-plane. The arc in the lower half-plane vanishes, as before, while the loop contour is clockwise (contributing an extra minus sign) and encloses the lower pole:

$$\oint dz \frac{e^{-iz}}{4z^2+1} = -2\pi i \,\text{Res}\left[\frac{e^{-iz}}{4z^2+1},\; z = -\frac{i}{2}\right] \tag{9.69}$$

$$= -2\pi i \frac{e^{-1/2}}{-4i} \tag{9.70}$$

$$= \frac{\pi}{2\sqrt{e}}. \tag{9.71}$$

Taking the real part, we obtain the same result as before.

9.5 Principal value integrals (optional topic)

Sometimes, we come across integrals that have poles lying on the desired integration contour.

As an example, consider

$$I = \int_{-\infty}^{\infty} dx\, \frac{\sin(x)}{x}. \tag{9.72}$$

Because of the series expansion of the sine function (see Section 1.2), the integrand does not diverge at $x = 0$, and the integral is in fact convergent. The integral can be solved without using complex numbers by using the arcane trick of differentiating under the integral sign (see Section 2.6). But it can also be solved via contour integration.

We start by writing

$$I = \text{Im}[I'], \quad \text{where} \quad I' = \int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}. \tag{9.73}$$

We want to calculate I' with the help of contour integration. But there's something strange about I': the complex integrand has a pole at $z = 0$, right on the real line!

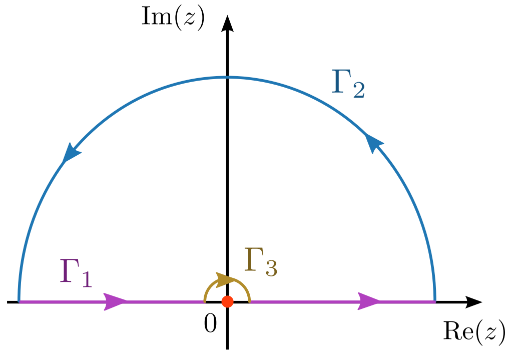
To handle this, we split I' into two integrals, one going over $-\infty < x < -\epsilon$ (where ϵ is some positive infinitesimal), and the other over $\epsilon < x < \infty$:

$$I' = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} dx\, \frac{e^{ix}}{x} + \int_{\epsilon}^{\infty} dx\, \frac{e^{ix}}{x}\right] \tag{9.74}$$

$$\equiv \mathcal{P}\int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}. \tag{9.75}$$

In the last line, the notation $\mathcal{P}[\cdots]$ is short-hand for this procedure of "chopping away" an infinitesimal segment surrounding the pole. This is called **taking the principal value** of the integral. (Note: even though this bears the same name as the previously-discussed "principal values" for multi-valued complex operations from Chapter 8, the two concepts are unrelated.)

Now consider the loop contour shown in the figure below. The loop follows the principal-value contour along the real axis, skips over the pole at $z = 0$ and arcs back along the upper half-plane. Since it encloses no poles, the loop integral vanishes by Cauchy's integral theorem.



This loop can be decomposed into several sub-contours:

1. Γ_1, consisting of the segments along the real axis.
2. Γ_2, the large counter-clockwise semi-circular arc.
3. Γ_3, the infinitesimal clockwise semi-circular arc that skips around $z = 0$.

The integral over Γ_1 is the principal-value integral we are interested in. The integral over Γ_2 vanishes by Jordan's lemma. The integral over Γ_3 can be calculated by parameterization:

$$\int_{\Gamma_3} dz\, \frac{e^{iz}}{z} = \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{i\epsilon\exp(i\theta)}}{\epsilon e^{i\theta}} \left(i\epsilon e^{i\theta}\right) d\theta \tag{9.76}$$

$$= i\int_{\pi}^{0} d\theta \tag{9.77}$$

$$= -i\pi. \tag{9.78}$$

Intutively, since encircling a pole anticlockwise gives a factor of $2\pi i$ times the residue (which is 1 in this case), a clockwise semi-circle is associated with a factor of $-i\pi$. Finally, putting everything together,

$$\underbrace{\int_{\Gamma_1+\Gamma_2+\Gamma_3} f(z)dz}_{=\,0\ \text{(Cauchy's integral theorem)}} = \underbrace{\int_{\Gamma_1} f(z)dz}_{=\,I'} + \underbrace{\int_{\Gamma_2} f(z)dz}_{=\,0\ \text{(Jordan's lemma)}} + \underbrace{\int_{\Gamma_3} f(z)dz}_{=\,-i\pi}. \tag{9.79}$$

Hence,

$$I = \text{Im}(I') = \text{Im}(i\pi) = \pi. \tag{9.80}$$

This agrees with the result obtained by the method of differentiating under the integral sign from Section 2.6.

Alternatively, we could have chosen the loop contour so that it skips *below* the pole at $z = 0$. In that case, the loop integral would be non-zero, and can be evaluated using the residue theorem. The final result is the same.

9.6 Exercises

1. Is the concept of a contour integral well-defined if the integrand $f(z)$ is non-differentiable along the contour? Why or why not?

2. In Section 9.4, we dealt with the integral
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}. \tag{9.81}$$
Redo this calculation, but this time close the contour in the lower half-plane. Show that the result is the same. [solution available]

3. Calculate
$$\int_{-\infty}^{\infty} dx\, \frac{1}{x^4+1}. \tag{9.82}$$

4. Calculate
$$\int_{-\infty}^{\infty} dx\, \left[\frac{\sin(x)}{x}\right]^2. \tag{9.83}$$

5. Calculate
$$\int_{0}^{\infty} dx \frac{x^\lambda}{x+1}, \quad \text{where } -1 < \lambda < 0. \tag{9.84}$$
Hint: place the integrand's branch cut along the positive real axis.

6. Solve the definite integral
$$I = \int_0^{2\pi} \frac{d\phi}{\cos\phi + 3}, \tag{9.85}$$
via the following steps. First, show that along a unit circle in the complex plane centered at the origin,
$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right), \tag{9.86}$$
where $z(\phi) = \exp(i\phi)$. Then define a complex function $f(z)$ such that the loop integral $\oint f(z)\, dz$, taken over the circular contour, is equal to I. Hence, calculate I.
[solution available]

7. Suppose $f(z)$ is analytic everywhere in the upper half-plane, including the real line, and that its magnitude vanishes as $1/|z|$ or faster as $|z| \to \infty$. Find the value of the principal-value integral
$$\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx\right], \tag{9.87}$$
where a is some real constant. Hence, prove that the real and imaginary parts of f along the real line are related by
$$\mathrm{Re}\big[f(x)\big] = \frac{1}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(w)]}{w-x} dw\right] \tag{9.88}$$
$$\mathrm{Im}\big[f(x)\big] = -\frac{1}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(w)]}{w-x} dw\right]. \tag{9.89}$$
These are called the **Kramers-Kronig relations**. In physics, these relations impose important constraints on the frequency dependence of the real and imaginary parts of the dielectric function (the square of the complex refractive index).
[solution available]