8. Mathematical Functions

When introducing complex algebra in Chapter 4, we postponed discussion of what it means to raise a complex number to a non-integer power, such as $z^{1/2}$, $z^{4/3}$, or z^{π}. It is now time to open that can of worms.

8.1 Non-integer powers as multi-valued operations

Given a complex number in its polar representation, $z = re^{i\theta}$, raising to the power of p could be handled this way:

$$z^p = \left(re^{i\theta}\right)^p = r^p e^{ip\theta}. \tag{8.1}$$

Let's take a closer look at the complex exponential term $e^{ip\theta}$. Since $\theta = \arg(z)$ is an angle, we can change it by multiples of 2π without altering the value of z. So we can re-write the above equation as

$$z^p = \left(r\, e^{i(\theta + 2\pi n)}\right)^p = \left(r^p e^{ip\theta}\right) e^{2\pi i n p}, \quad \text{where } n \in \mathbb{Z}. \tag{8.2}$$

If p is an integer, this re-statement is somewhat trivial: no matter what integer n we take, $2\pi n p$ is a multiple of 2π, so z^p ends up with the same value:

$$z^p = r^p e^{ip\theta} \text{ unambiguously } (\text{if } p \in \mathbb{Z}). \tag{8.3}$$

But if p is not an integer, the $\exp(2\pi i n p)$ factor takes on different values for different n. In that case, the "power of p" is a **multi-valued operation**. It cannot be treated as a function in the usual sense, since functions must have unambiguous outputs.

8.2 Roots of unity

Let's take a closer look at the problematic exponential term,

$$\exp(2\pi i n p), \quad \text{where } n \in \mathbb{Z}. \tag{8.4}$$

If p is irrational, $2\pi n p$ never repeats modulo 2π. Thus, z^p has an infinite set of values, one for each integer n.

More interesting is the case of a non-integer *rational* power. Any rational number can be written as $p = P/Q$ where P and Q are integers with no common divisor. It can be proven using modular arithmetic (though we will not go into the details) that $2\pi n\,(P/Q)$ has exactly Q unique values modulo 2π:

$$2\pi n p = 2\pi n \left(\frac{P}{Q}\right) = 2\pi \times \left\{0, \frac{1}{Q}, \frac{2}{Q}, \ldots, \frac{(Q-1)}{Q}\right\} \quad (\text{modulo } 2\pi). \tag{8.5}$$

The set of values is independent of the numerator P. The value of P merely affects the sequence in which the numbers are generated as we step though the integer values of n. This is demonstrated by the following examples:

Example—Consider the complex square root operation, $z^{1/2}$. If we write z in its polar respresentation, $z = re^{i\theta}$, then

$$z^{1/2} = \left[r\, e^{i(\theta + 2\pi n)}\right]^{1/2} = r^{1/2}\, e^{i\theta/2}\, e^{i\pi n}, \quad n \in \mathbb{Z}. \tag{8.6}$$

The $e^{i\pi n}$ factor has two possible values: $+1$ (for even n) and -1 (for odd n). Hence, the values of the square root are

$$z^{1/2} = r^{1/2}\, e^{i\theta/2} \,\times\, \{1, -1\}. \tag{8.7}$$

Example—Consider the cube root operation $z^{1/3}$. Taking $z = re^{i\theta}$, we obtain

$$z^{1/3} = r^{1/3}\, e^{i\theta/3}\, e^{2\pi in/3}, \quad n \in \mathbb{Z}. \tag{8.8}$$

Running through n gives

n	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$2\pi n/3$	$\cdots$	$-4\pi/3$	$-2\pi/3$	0	$2\pi/3$	$4\pi/3$	$6\pi/3$	$8\pi/3$	$\cdots$
$e^{2\pi in/3}$	$\cdots$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$\cdots$

Hence, the cube root operation has three distinct values:

$$z^{1/3} = r^{1/3}\, e^{i\theta/3} \times \left\{ 1,\ e^{2\pi i/3},\ e^{-2\pi i/3} \right\}. \tag{8.9}$$

Example—Consider the operation $z^{2/3}$. Again taking $z = re^{i\theta}$,

$$z^{2/3} = r^{2/3}\, e^{2i\theta/3}\, e^{4\pi in/3}, \quad n \in \mathbb{Z}. \tag{8.10}$$

Running through n gives

| n	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$4\pi n/3$	$\cdots$	$-8\pi/3$	$-4\pi/3$	0	$4\pi/3$	$8\pi/3$	$12\pi/3$	$16\pi/3$	$\cdots$
$e^{4\pi in/3}$	$\cdots$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$\cdots$

Hence,

$$z^{2/3} = r^{2/3}\, e^{2i\theta/3} \times \left\{ 1,\ e^{2\pi i/3},\ e^{-2\pi i/3} \right\}. \tag{8.11}$$

Note that the set of values in curly brackets is the same as in the previous example, demonstrating that the numerator P does not affect the set.

From the above examples, we deduce the following expression for rational powers:

$$z^{P/Q} = r^{P/Q}\, e^{i\theta\,(P/Q)} \times \left\{ 1,\ e^{2\pi i/Q},\ e^{4\pi i/Q},\ \ldots, e^{2\pi i(1-Q)/Q} \right\}. \tag{8.12}$$

The quantities in the curly brackets are called the **roots of unity**. In the complex plane, they sit at Q evenly-spaced points on the unit circle, with 1 as one of the values:
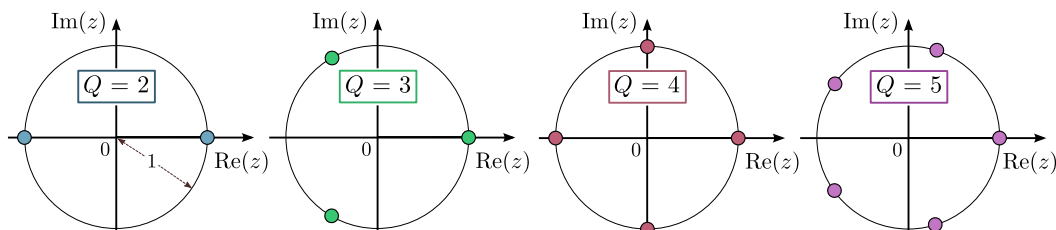
8.3 Complex logarithms

Here is another way to think about non-integer powers. Recall what it means to raise a number to, say, the power of 5: we simply multiply the number by itself five times. What about raising a number to a non-integer power p? For the real case, the power is defined as a combination of exponential and logarithm functions, as we saw in Section 0.2:

$$x^p \equiv \exp\big[\, p \ln(x)\big]. \tag{8.13}$$

This definition relies on the fact that, for real inputs, the logarithm is a well-defined function. That, in turn, comes from the definition of the logarithm as the inverse of the exponential function. Since the real exponential is one-to-one, its inverse is also one-to-one.

The complex exponential, however, is many-to-one, since changing its input by any multiple of $2\pi i$ yields the same output:

$$\exp(z + 2\pi i n) = \exp(z) \cdot e^{2\pi i n} = \exp(z) \quad \text{for all } n \in \mathbb{Z}. \tag{8.14}$$

The inverse of the complex exponential is the **complex logarithm**. Since the complex exponential is many-to-one, the complex logarithm is one-to-many. For each z, there is an infinite set of values for $\ln(z)$, separated by integer multiples of $2\pi i$:

$$\ln(z) = \big[\ln(z)\big]_{\text{p.v.}} + 2\pi i n, \quad n \in \mathbb{Z}. \tag{8.15}$$

Here, $[\ln(z)]_{\text{p.v.}}$ denotes the **principal value** of $\ln(z)$, which refers to a reference value of the logarithm operation (which we'll define later). Do not think of the principal value as the "actual" result of the $\ln(z)$ operation! There are multiple values, each equally legitimate; the principal value is merely one of them.

We now apply the formula $z^p \equiv \exp[p \ln(z)]$, with $\ln(z)$ as the multi-valued complex logarithm. Then

$$z^p = \exp\Big\{ p\big(\big[\ln(z)\big]_{\text{p.v.}} + 2\pi i n\big)\Big\} \tag{8.16}$$

$$= \exp\Big\{ p\big[\ln(z)\big]_{\text{p.v.}}\Big\} e^{2\pi i n p}, \quad n \in \mathbb{Z} \tag{8.17}$$

$$= \big[z^p\big]_{\text{p.v.}} e^{2\pi i n p}, \qquad\qquad n \in \mathbb{Z}. \tag{8.18}$$

The factor of $e^{2\pi i n p}$, which is responsible for the multi-valuedness, corresponds to the roots of unity discussed in Section 8.2.

8.4 Branches

We have discussed two examples of multi-valued complex operations: non-integer powers and the complex logarithm. However, we usually prefer to deal with functions rather than multi-valued operations. One reason is that the concept of the complex derivative is based on functions, not multi-valued operations.

There is a standard procedure to convert multi-valued operations into functions. First, we define one or more curve(s) in the complex plane, called **branch cuts** (the reason for this name will be explained later). Next, we modify the domain (i.e., the set of permissible inputs) by excluding all values of z lying on a branch cut. Then the outputs of the multi-valued operation can be grouped into discrete **branches**, each behaving as a function.

The above procedure can be understood through the example of the square root.

8.4.1 Branches of the complex square root

We saw in Section 8.2 that the complex square root, $z^{1/2}$, has two possible values. We can define the two branches as follows:

1. Define a branch cut along the negative real axis, so that the domain excludes all z along the branch cut. In other words, we will only consider complex numbers whose polar representation can be written as

$$z = re^{i\theta}, \quad \theta \in (-\pi, \pi).$$

(For those unfamiliar with this notation, $\theta \in (-\pi, \pi)$ refers to the interval $-\pi < \theta < \pi$. The parentheses indicate that the boundary values of $-\pi$ and π are excluded. By contrast, we would write $\theta \in [-\pi, \pi]$ to refer to the interval $-\pi \leq \theta \leq \pi$, with the square brackets indicating that the boundary values are included.)
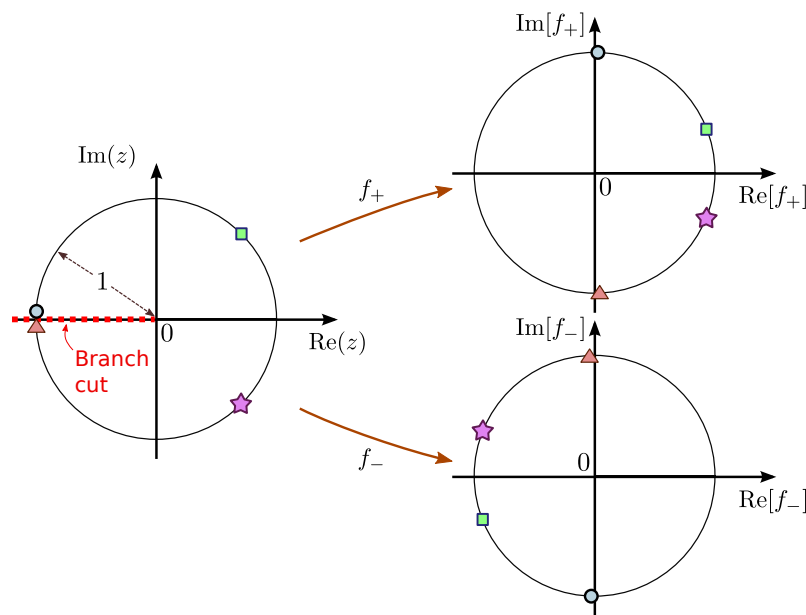
2. One branch is associated with the root of unity $+1$. On this branch, for $z = re^{i\theta}$, the value is
$$f_+(z) = r^{1/2}\, e^{i\theta/2}, \quad \theta \in (-\pi, \pi).$$

3. The other branch is associated with the root of unity -1. On this branch, the value is

$$f_-(z) = -r^{1/2}\, e^{i\theta/2}, \quad \theta \in (-\pi, \pi).$$

The following plot shows how varying z affects the positions of $f_+(z)$ and $f_-(z)$ in the complex plane:

In the left subplot, the red dashes indicate the branch cut, and the various symbols (circle, square, star, and triangle) indicate representative values of z. In the right subplots, the symbols indicate the corresponding positions of $f_+(z)$ and $f_-(z)$ in the complex plane.

Note that $f_+(z)$ always lies in the right half of the complex plane, whereas $f_-(z)$ lies in the left half of the complex plane. Both f_+ and f_- are well-defined functions with unambiguous outputs, albeit with domains that do not cover the entire complex plane (i.e., the branch cut is excluded).

It can moreover be shown that these functions are analytic over all of the complex plane except the branch cut (see Section 6.2); this can be proven using the Cauchy-Riemann equations, and is left as an exercise.

The end-point of the branch cut is called a **branch point**. For $z = 0$, both branches give the same result: $f_+(0) = f_-(0) = 0$. We will have more to say about branch points in Section 8.4.3.

### 8.4.2   Different branch cuts for the complex square root

You may be wondering why the branch cut has to lie along the negative real axis. In fact, this choice is not unique. For instance, we could place the branch cut along the positive real axis. This corresponds to specifying the input z using a different interval for θ:

$$z = re^{i\theta}, \quad \theta \in (0, 2\pi). \tag{8.19}$$

Next, we use the same formulas as before to define the branches of the complex square root:

$$f_\pm(z) = \pm r^{1/2}\, e^{i\theta/2}. \tag{8.20}$$

But because the domain of θ has been changed to $(0, 2\pi)$, the set of inputs z now excludes the positive real axis. With this new choice of branch cut, the values produced by the branch functions are shown in the following figure:
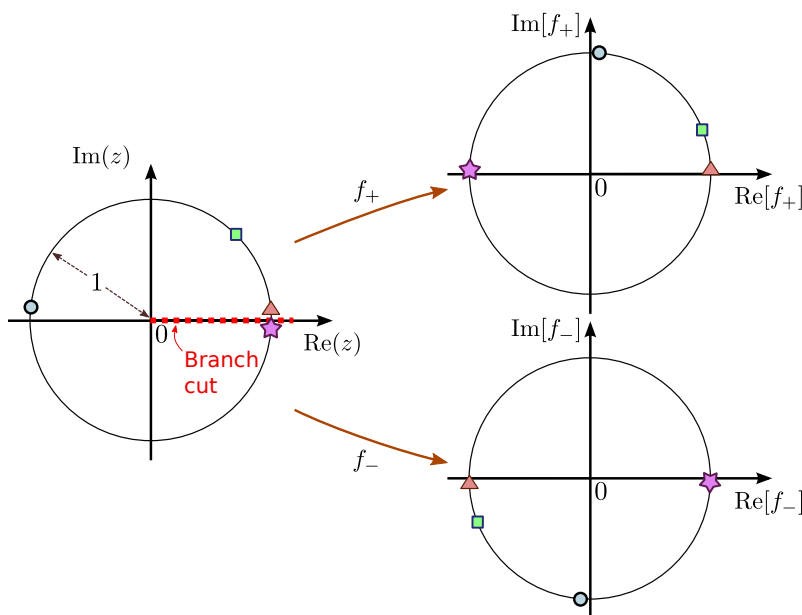

The two branch functions are different from what we had before. Now, $f_+(z)$ is always in the upper half of the complex plane, and $f_-(z)$ in the lower half of the complex plane. However, both branches still have the same value at the branch point: $f_+(0) = f_-(0) = 0$.

We can think of the branch cut as a boundary where two branches are "glued" together, so that crossing the branch cut brings us from one branch to a different branch. For example, in the left subplot, consider the value of z indicated by the triangle, which lies just above the branch cut. In the right subplots, observe that the corresponding value of $f_+(z)$ lies just above the positive real axis, and $f_-(z)$ lies just below the negative real axis.

Next, consider the value of z indicated by the star, which lies just below the branch cut. Going from the triangle to the star is equivalent to a small downwards displacement of z, "crossing" the branch cut. Now the values of the positive and negative branches are swapped: $f_-(z)$ lies just below the positive real axis, near where $f_+(z)$ was previously, and $f_+(z)$ now lies just above the negative real axis where $f_-(z)$ was previously.

The three-dimensional plot below provides another way to visualize the role of the branch cut. Here, the horizontal axes correspond to $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. The vertical axis shows the arguments for the two values of the complex square root, with $\arg\big[f_+(z)\big]$ plotted in orange and $\arg\big[f_-(z)\big]$ plotted in blue. The choice of branch cut, shown as a red line, is just a choice about how to divide up the branches of a multi-valued operation.
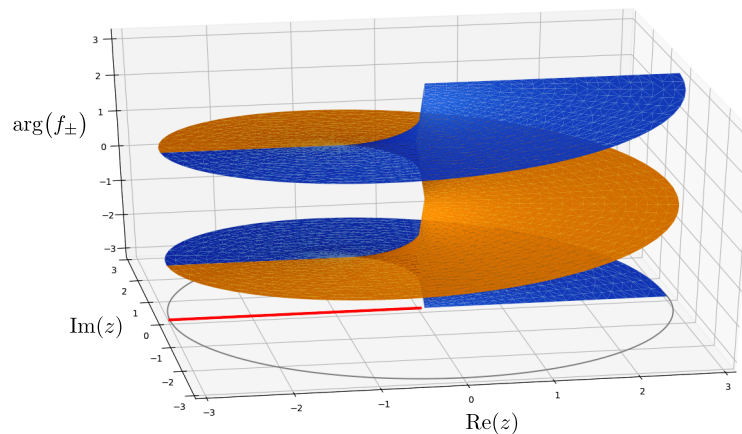
8.4.3   Branch points

The tip of each branch cut is called a **branch point**. A branch point is a point where different branches have the same value. Whereas the choice of branch cuts is non-unique, the branch points of a multi-valued operation are uniquely determined.

For the purposes of this course, you mostly only need to remember the branch points arising from two common cases:

- The z^p operation (for non-integer p) has branch points at $z = 0$ and $z = \infty$.

- The complex logarithm has branch points at $z = 0$ and $z = \infty$.

We can easily see that z^p must have a branch point at $z = 0$: at this point, the value has to be 0, regardless of the choice of root of unity. As for the branch point at $z = \infty$, understanding it requires us to know more about the concept of "infinity" for complex numbers.

8.5   The meaning of "infinity" for complex numbers (optional topic)

When talking about the **complex infinity**, $z = \infty$, we are referring to a complex number with infinite magnitude and *undefined* argument.

The idea of a complex number having undefined argument may seem strange, but actually we already know of another complex number with this feature: $z = 0$ has zero magnitude and undefined argument. These two special complex numbers are the reciprocals of each other: $1/\infty = 0$ and $1/0 = \infty$.

The complex ∞ behaves differently from the familiar concept of infinity associated with real numbers. For real numbers, positive infinity ($+\infty$) is distinct from negative infinity ($-\infty$). But this doesn't hold for complex numbers, since complex numbers occupy a two-dimensional plane rather than a line. Thus, for complex numbers it does not make sense to define "positive infinity" and "negative infinity" as distinct entities.

From this, we can see why z^p has a branch point at $z = \infty$. For any finite and nonzero z, we can write $z = re^{i\theta}$, where r is a positive number. The values of the z^p operation have the form

$$r^p\, e^{ip\theta} \,\times\, \{\text{roots of unity}\}. \tag{8.21}$$

Regardless of which root of unity we pick, the magnitude is r^p; as $r \to \infty$, the magnitude goes to infinity and the overall complex value goes to ∞. Hence, at $z = \infty$ all the branches of z^p have the same value (i.e., ∞).

By similar reasoning, one can prove that $\ln(z)$ has branch points at $z = 0$ and $z = \infty$. This is left as an exercise.

8.6   Branch cuts for general multi-valued operations

Having discussed the simplest multi-valued operations, z^p and $\ln(z)$, here is how to assign branch cuts for more general multi-valued operations. This is a two-step process:

1. Locate the branch points.

2. Assign branch cuts in the complex plane, such that (i) every branch point has a branch cut ending on it, and (ii) every branch cut ends on a branch point. The branch cuts should not intersect.

The choice of where to place branch cuts is not unique. Branch cuts are usually chosen to be straight lines, for simplicity, but this is not necessary. Different choices of branch cuts correspond to different ways of partitioning the values of the multi-valued operation into separate branches.

8.6.1   An important example

We can illustrate the process of assigning branch cuts, and defining branch functions, with the following multi-valued operation:

$$f(z) = \ln\left(\frac{z+1}{z-1}\right). \tag{8.22}$$

This is multi-valued because of the presence of the complex logarithm. The branch points are $z = 1$ and $z = -1$, as these are the points where the input to the logarithm becomes ∞ or 0 respectively. Note that $z = \infty$ is *not* a branch point; at $z = \infty$, the input to the logarithm is -1, which is not a branch point for the logarithm.

We can assign any branch cut that joins the branch points at $z = \pm 1$. A convenient choice is shown below:
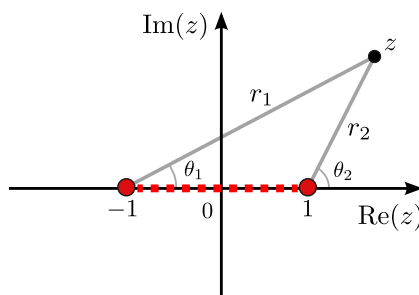
This choice of branch cut is nice because we can express the $z + 1$ and $z - 1$ terms using the polar representations

$$z + 1 = r_1\, e^{i\theta_1}, \tag{8.23}$$
$$z - 1 = r_2\, e^{i\theta_2}, \tag{8.24}$$

where r_1, r_2, θ_1, and θ_2 are shown graphically in the above figure. The positioning of the branch cut corresponds to a particular choice for the ranges of the complex arguments θ_1 and θ_2. As we'll shortly see, the present choice of branch cut corresponds to

$$\theta_1 \in (-\pi, \pi), \quad \theta_2 \in (-\pi, \pi). \tag{8.25}$$

Hence, $f(z)$ can be written as

$$f(z) = \ln\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2 + 2\pi m), \quad \text{where} \quad \begin{cases} m \in \mathbb{Z}, \\ z = -1 + r_1\, e^{i\theta_1} = 1 + r_2\, e^{i\theta_2} \\ \theta_1, \theta_2 \in (-\pi, \pi). \end{cases} \tag{8.26}$$

The choice of m specifies the branch, and we can choose $m = 0$ as the principal branch.

Let us verify that setting $\theta_1 \in (-\pi, \pi)$ and $\theta_2 \in (-\pi, \pi)$ is consistent with our choice of branch cut. Consider the principal branch, and compare the outputs of the above formula for z just above the real axis, and for z just below the real axis. There are three cases of interest, depending on $\mathrm{Re}[z]$:

Firstly, for $\mathrm{Re}[z] < -1$ (to the left of the leftmost branch point),

$$\mathrm{Im}[z] = 0^+ \Rightarrow \theta_1 \to \pi, \quad \theta_2 \to \pi \quad \Rightarrow f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((\pi) - (\pi)\big) \qquad = \ln\left(\frac{r_1}{r_2}\right) \tag{8.27}$$

$$\mathrm{Im}[z] = 0^- \Rightarrow \theta_1 \to -\pi, \theta_2 \to -\pi \Rightarrow f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((-\pi) - (-\pi)\big) = \ln\left(\frac{r_1}{r_2}\right) \tag{8.28}$$

(If you're not sure why θ_1 and θ_2 have these values, look carefully at the above figure, and think about what values θ_1 and θ_2 would have for, say, $z = -2 + 0.001i$ or $z = -2 - 0.001i$.) Thus, there is no discontinuity along this segment of the real axis.

Secondly, for $-1 < \mathrm{Re}[z] < 1$ (between the two branch points),

$$\mathrm{Im}[z] = 0^+ \Rightarrow \theta_1 \to 0, \theta_2 \to \pi \quad \Rightarrow f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((0) - (\pi)\big) \quad = \ln\left(\frac{r_1}{r_2}\right) - i\pi \tag{8.29}$$

$$\mathrm{Im}[z] = 0^- \Rightarrow \theta_1 \to 0, \theta_2 \to -\pi \Rightarrow f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((0) - (-\pi)\big) = \ln\left(\frac{r_1}{r_2}\right) + i\pi \tag{8.30}$$

Hence, in the segment between the two branch points, there is a discontinuity of $\pm 2\pi i$ on different sides of the real axis. The value of this discontinuity is exactly equal, of course, to the separation between the different branches of the complex logarithm.

Finally, for $\mathrm{Re}[z] > 1$ (to the right of the rightmost branch point), there is again no discontinuity:

$$\mathrm{Im}[z] = 0^+ \Rightarrow \theta_1 \to 0, \theta_2 \to 0 \quad \Rightarrow \quad f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((0) - (0)\big) = \ln\left(\frac{r_1}{r_2}\right) \tag{8.31}$$

$$\mathrm{Im}[z] = 0^- \Rightarrow \theta_1 \to 0, \theta_2 \to 0 \quad \Rightarrow \quad f(z) = \ln\left(\frac{r_1}{r_2}\right) + i\big((0) - (0)\big) = \ln\left(\frac{r_1}{r_2}\right). \tag{8.32}$$

8.7   Exercises

1. Find the values of $(i)^i$.   [solution available]

2. Prove that $\ln(z)$ has branch points at $z = 0$ and $z = \infty$.   [solution available]

3. For each of the following multi-valued functions, find all the possible function values, at the specified z:

   (a) $z^{1/3}$ at $z = 1$.
   (b) $z^{3/5}$ at $z = i$.
   (c) $\ln(z + i)$ at $z = 1$.
   (d) $\cos^{-1}(z)$ at $z = i$

4. For the square root operation $z^{1/2}$, choose a branch cut. Then show that both the branch functions $f_\pm(z)$ are analytic over all of $\mathbb{C}$ excluding the branch cut.

5. Consider $f(z) = \ln(z + a) - \ln(z - a)$. For simplicity, let a be a positive real number. As discussed in Section 8.6.1, we can write this as

$$f(z) = \ln\left|\frac{z + a}{z - a}\right| + i(\theta_+ - \theta_-), \qquad \theta_\pm \equiv \arg(z \pm a). \tag{8.33}$$

Suppose we represent the arguments as $\theta_+ \in (-\pi, \pi)$ and $\theta_- \in (-\pi, \pi)$. Explain why this implies a branch cut consisting of a straight line joining a with $-a$. Using this representation, calculate the change in $f(z)$ over an infinitesimal loop encircling $z = a$ or $z = -a$. Calculate also the change in $f(z)$ over a loop of radius $R \gg a$ encircling the origin (and thus enclosing both branch points).