4. Complex Oscillations

The most common use of complex numbers in physics is for analyzing oscillations and waves. We will illustrate this with a simple but crucially important model, the damped harmonic oscillator.

4.1 The harmonic oscillator equation

The damped harmonic oscillator describes a mechanical system consisting of a particle of mass $m$, subject to a spring force and a damping force:

\[
\begin{align*}
\text{Spring force} & \quad F_1 = -m\omega_0^2 x \\
\text{Damping force} & \quad F_2 = -2m\gamma \dot{x}
\end{align*}
\]

The particle can move along one dimension, and we let $x(t)$ denote its displacement from the origin. The damping coefficient is $2m\gamma$, and the spring constant is $k = m\omega_0^2$. The parameters $m$, $\gamma$, and $\omega_0$ are all positive real numbers. (The quantity $\omega_0$ is called the “natural frequency of oscillation”, because in the absence of the damping force this system would act as a simple harmonic oscillator with frequency $\omega_0$.)

The motion of the particle is described by Newton’s second law:

\[
m \frac{d^2 x}{dt^2} = F(x, t) = -2m\gamma \frac{dx}{dt} - m\omega_0^2 x(t).
\]  

(1)

Dividing by the common factor of $m$, and bringing everything to one side, gives

\[
\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x(t) = 0.
\]

(2)

We call this ordinary differential equation the damped harmonic oscillator equation. Since it’s a second-order ordinary differential equation (ODE), the general solution must contain two independent parameters. If we state the initial displacement and velocity, $x(0)$ and $\dot{x}(0)$, there is a unique specific solution.

\[
\text{Note}
\]

Sometimes, we write the damped harmonic oscillator equation a bit differently:

\[
\left[\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right] x(t) = 0.
\]  

(3)

The quantity in the square brackets is regarded as an operator acting on $x(t)$. This operator consists of the sum of three terms: a second-derivative operator, a constant times a first derivative, and multiplication by a constant.

We are interested in solving for $x(t)$. For the simple (undamped) harmonic oscillator, which is the case where $\gamma = 0$, we know what the general solution looks like:

\[
x(t) = x_0 \cos(\omega_0 t + \phi).
\]

(4)

The particle oscillates around the equilibrium position, $x = 0$, because the spring force keeps pushing it towards the origin and its momentum causes it to “overshoot”. For the damped
harmonic oscillator ($\gamma > 0$), however, the damping force causes the particle to lose energy during its motion, so that as $t \to \infty$, both $x$ and $\dot{x}$ go to zero. A typical solution is shown in the following figure:

![Graph of a typical solution to the damped harmonic oscillator equation.](image)

### 4.2 Complex solution

The variable $x(t)$ stands for the displacement of a particle, which is a real quantity. But in order to solve the damped harmonic oscillator equation, it’s useful if we generalize $x(t)$ to complex values. In other words, let’s treat the harmonic oscillator equation as a complex ODE:

$$\frac{d^2 z}{dt^2} + 2\gamma \frac{dz}{dt} + \omega_0^2 z(t) = 0, \quad z(t) \in \mathbb{C}. \quad (5)$$

The parameter-counting rule that we discussed for real ODEs can also be applied to complex ODEs, except that we use complex parameters in place of real parameters. In this case, the complex damped harmonic oscillator equation is a second-order ODE, so its general solution should contain two independent complex parameters.

Once we have that general solution, we can do one of two things: (i) plug in a complete set of (real) boundary conditions, which will give a real specific solution, or (ii) take the real part of the complex general solution, which will give the general solution to the real differential equation. We will discuss these two approaches later; for the moment, let’s focus on finding the solution to the complex ODE.

To find the complex solution, first note that the equation is linear. This means that if we have two solutions $z_1(t)$ and $z_2(t)$, then any combination

$$\alpha z_1(t) + \beta z_2(t), \quad \text{where } \alpha, \beta \in \mathbb{C} \quad (6)$$

is also a solution. Therefore, a good strategy is to find several specific solutions, and then combine them linearly to form a more general solution. We simply make a guess (or an ansatz) for a specific solution:

$$z(t) = e^{-i\omega t}, \quad (7)$$

where $\omega$ is a constant to be determined (it could be complex). The first and second derivatives of Eq. (7) are:

$$\frac{dz}{dt} = -i\omega e^{-i\omega t} \quad (8)$$

$$\frac{d^2 z}{dt^2} = -\omega^2 e^{-i\omega t} \quad (9)$$

Substituting these into the differential equation (5) gives:

$$[-\omega^2 - 2i\gamma \omega + \omega_0^2] e^{-i\omega t} = 0. \quad (10)$$
This equation can be satisfied for all $t$ if the complex quadratic on the left side is zero:
\[ -\omega^2 - 2i\gamma \omega + \omega_0^2 = 0. \]  
(11)

In other words, we need values of $\omega$ which solve this quadratic equation. The solutions can be obtained from the quadratic formula:
\[ \omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \]  
(12)

Hence, we arrive at solutions which are oscillations with complex frequencies:
\[ z(t) = \exp\left(-i\omega_{\pm} t\right), \quad \text{where} \quad \omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \]  
(13)

For each value of $\gamma$ and $\omega_0$, there are two possible frequencies, $\omega_+$ and $\omega_-$. For either choice of complex frequency, the above expression for $z(t)$ gives a valid specific solution for the complex damped harmonic oscillator equation.

4.2.1 Complex frequencies

What does it mean to have an oscillation with a complex frequency? If we write the real and imaginary parts of the frequency as $\omega = \omega_R + i\omega_I$, then
\[ z(t) = e^{-i\omega_I t} = e^{\omega_I t} e^{-i\omega_R t}. \]  
(14)

If both $\omega_R$ and $\omega_I$ are non-zero, this describes a spiral trajectory in the complex plane, whose magnitude is either increasing or decreasing with time, depending on the sign of $\omega_O$. This is because we can write
\[ z(t) = e^{\omega_I t} e^{-i\omega_R t} = R(t) e^{i\theta(t)}, \quad \text{where} \quad R(t) = e^{\omega_I t}, \quad \theta(t) = -\omega_R t. \]  
(15)

We therefore conclude that the real part of $\omega$ determines the (angular) frequency of oscillation, whereas the imaginary part determines whether the oscillation amplitude is either growing with time (amplification) or shrinking with time (damping). A positive imaginary part implies amplification, and a negative imaginary part implies damping, while zero imaginary part (i.e., a real frequency) implies constant-amplitude oscillation.

Now let’s look at the damped harmonic oscillator’s complex frequencies:
\[ \omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \]  
(16)

These depend on two real parameters: $\gamma$ and $\omega_0$. In the plot below, you can see how the position of $\omega_{\pm}$ in the complex plane depends on the values of these parameters.
In particular, note the following features:

- For $\gamma = 0$ (zero damping), the two frequencies are both real, and take the values $\pm \omega_0$. This corresponds to undamped (or “simple”) harmonic oscillation at the oscillator’s natural frequency.
- If we increase $\gamma$ from zero with $\omega_0$ fixed, both $\omega_+ + \omega_-$ move downwards in the complex plane, along a circular arc. Because the imaginary part of the frequencies are negative, this implies damped oscillation.
- At $\gamma = \omega_0$, the frequencies meet along the imaginary axis.
- For $\gamma > \omega_0$, the two frequencies move apart along the imaginary axis. Purely imaginary frequencies correspond to a trajectory that simply decays without oscillating.

4.3 General solution for the damped harmonic oscillator

For now, suppose $\omega_0 \neq \gamma$. In the previous section, we found two classes of specific solutions, with complex frequencies $\omega_+$ and $\omega_-:

$$z_+(t) = e^{-i\omega_+ t} \quad \text{and} \quad z_-(t) = e^{-i\omega_- t}, \quad \omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \quad (17)$$

We can write down a more general solution consisting of a linear superposition of these specific solutions:

$$z(t) = \psi_+ e^{-i\omega_+ t} + \psi_- e^{-i\omega_- t} \quad (18)$$

$$= \psi_+ \exp \left[ \left( -\gamma - i\sqrt{\omega_0^2 - \gamma^2} \right) t \right] + \psi_- \exp \left[ \left( -\gamma + i\sqrt{\omega_0^2 - \gamma^2} \right) t \right]. \quad (19)$$

This contains two undetermined complex parameters, $\psi_+$ and $\psi_-$. These are independent parameters since they are the coefficients that multiply different functions (the functions are different because $\omega_0 \neq \gamma$ implies that $\omega_+ \neq \omega_-$. Hence, the above equation for $z(t)$ is a general solution for the complex damped harmonic oscillator equation.

To obtain the general solution to the real damped harmonic oscillator equation, we have to take the real part of the complex solution. The result can be further simplified depending on whether $\omega_0^2 - \gamma^2$ is positive or negative. This leads to what are called under-damped solutions and over-damped solutions, to be discussed in the following subsections.

Now that we have $\omega_0 = \gamma$? In this instance, $\omega_+ = \omega_-$, which means that $\psi_+$ and $\psi_-$ aren’t independent parameters. Therefore, the above equation for $z(t)$ isn’t a valid general solution in this particular case! Instead, the general solution is something called a critically-damped solution, which we will discuss in Section 4.3.3.

4.3.1 Under-damped motion

For $\omega_0 > \gamma$, let us define, for convenience,

$$\Omega = \sqrt{\omega_0^2 - \gamma^2}. \quad (20)$$

Then we can simplify the real solution as follows:

$$z(t) = \text{Re} \left[ z(t) \right] = e^{-\gamma t} \text{Re} \left[ \psi_+ e^{-i\Omega t} + \psi_- e^{i\Omega t} \right] \quad (21)$$

$$= e^{-\gamma t} \left[ A \cos (\Omega t) + B \sin (\Omega t) \right], \quad \text{where} \quad A, B \in \mathbb{R} \quad (23)$$
With a bit of algebra, we can show that

\[ A = \text{Re} [\psi_+ + \psi_-], \quad B = \text{Im} [\psi_+ - \psi_-]. \]  

(24)

The coefficients \( A \) and \( B \) act as two independent real parameters, so this is a valid general solution for the real damped harmonic oscillator equation. Using the trigonometric formulas, the solution can be equivalently written as

\[ x(t) = Ce^{-\gamma t} \cos [\Omega t + \Phi], \]  

(25)

with the parameters \( C = \sqrt{A^2 + B^2} \) and \( \Phi = -\tan^{-1} [B/A] \).

Either way, this is called an under-damped solution. A typical graph is shown below:

The particle undergoes oscillation, but the amplitude of oscillation decreases with time. The decrease in the amplitude can be visualized using a smooth “envelope” given by \( \pm Ce^{-\gamma t} \), which is drawn with dashes in the figure. Inside this envelope, the trajectory oscillates with frequency \( \Omega = \sqrt{\omega_0^2 - \gamma^2} \), which is slightly less than the natural frequency of oscillation \( \omega_0 \).

### 4.3.2 Over-damped motion

For \( \omega_0 < \gamma \), the square root term becomes imaginary. It is convenient to define

\[ \Gamma = \sqrt{\gamma^2 - \omega_0^2} \Rightarrow \sqrt{\omega_0^2 - \gamma^2} = i\Gamma. \]  

(26)

Then the real solution simplifies in a different way:

\[ x(t) = \text{Re} [z(t)] \]  

(27)

\[ = \text{Re} \left[ \psi_+ e^{(-\gamma + \Gamma)t} + \psi_- e^{(-\gamma - \Gamma)t} \right] \]  

(28)

\[ = C_+ e^{-(\gamma - \Gamma)t} + C_- e^{-(\gamma + \Gamma)t}, \]  

(29)

where

\[ C_{\pm} = \text{Re} [\psi_{\pm}]. \]  

(30)

This is called an over-damped solution. The solution consists of two terms, both exponentially decaying in time, with \((\gamma - \Gamma)\) and \((\gamma + \Gamma)\) serving as the decay rates. Note that both decay rates are positive real numbers, because \( \Gamma < \gamma \) from the definition of \( \Gamma \). Also, note that the first decay rate \((\gamma - \Gamma)\) is a decreasing function of \( \gamma \), whereas the second decay rate \((\gamma + \Gamma)\) is an increasing function of \( \gamma \).
The larger decay rate, \((\gamma+\Gamma)\), is associated with a faster-decaying exponential. Therefore, at long times the second term becomes negligible compared to the first term. Then the solution approaches the limit

\[
x(t) \approx C_+ e^{-(\gamma-\Gamma)t} \quad \text{(large } t) .
\]

This limiting curve is shown as a red dash in the figure below.

This has an interesting implication: the stronger the damping, the slower the effective decay rate at long times. Why does this happen? In the over-damped regime, the motion of the oscillator is dominated by the damping force rather than the spring force; as the oscillator tries to return to its equilibrium position \(x = 0\), the damping acts against this motion. Hence, the stronger the damping, the slower the decay to equilibrium.

This contrasts sharply with the previously-discussed under-damped regime, where the spring force dominates the damping force. There, stronger damping speeds up the decay to equilibrium, by causing the kinetic energy of the oscillation to be dissipated more rapidly.

4.3.3 Critical damping

Critical damping occurs when \(\omega_0 = \gamma\). Under this special condition, the solution we previously derived reduces to

\[
z(t) = (\psi_+ + \psi_-) e^{-\gamma t}.
\]

This has only one independent complex parameter, i.e. the parameter \((\psi_+ + \psi_-)\). Therefore, it cannot be a general solution for the complex damped harmonic oscillator equation, which is still a second-order ODE.

We will not go into detail here regarding the procedure for finding the general solution for the critically-damped oscillator, leaving it as an exercise for the interested reader. Basically, we can Taylor expand the solution on either side of the critical point, and then show that there is a solution of the form

\[
z(t) = (A + Bt) e^{-\gamma t},
\]

which contains the desired two independent parameters.

The critically-damped solution contains an exponential decay constant of \(\gamma\), which is the same as the decay constant for the envelope function in the under-damped regime, and smaller than the (long-time) decay constants in the over-damped regime. Hence, we can regard the critically-damped solution as the fastest-decaying non-oscillatory solution.

This feature of critical damping is employed in many engineering contexts, the most familiar being automatic door closers. If the damping is too weak or the spring force is too strong (under-damped), the door will tend to slam shut, whereas if the damping is too strong or the spring force is too weak (under-damping), the door will take unnecessarily long to close. Hence, door closers need to be tuned to a “sweet spot” that corresponds to the critical damping point.
4.4 Stating the solution in terms of initial conditions

The general solution for the complex damped harmonic oscillator equation contains two undetermined parameters which are the complex amplitudes of the “clockwise” and “counter-clockwise” complex oscillations:

\[ z(t) = \psi_+ e^{-i\omega_+ t} + \psi_- e^{-i\omega_- t}, \quad \text{where} \quad \omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \]  \hspace{1cm} (34)

However, mechanics problems are often expressed in terms of an initial-value problem, which expresses the state of the system at some initial time \( t = 0 \). Suppose we are given \( z(0) \equiv x_0 \) and \( \dot{z}(0) \equiv v_0 \); then what is \( z(t) \) in terms of \( x_0 \) and \( v_0 \)?

We can solve the initial-value problem by finding \( z(0) \) and \( \dot{z}(0) \) in terms of the above general solution for \( z(t) \). The results are

\[ z(0) = \psi_+ + \psi_- = x_0 \]  \hspace{1cm} (35)

\[ \dot{z}(0) = -i\omega_+ \psi_+ - i\omega_- \psi_- = v_0. \]  \hspace{1cm} (36)

These two equations can be combined into a 2 \( \times \) 2 matrix equation:

\[
\begin{bmatrix}
1 & 1 \\
-i\omega_+ & -i\omega_-
\end{bmatrix}
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix}
= \begin{bmatrix}
x_0 \\
v_0
\end{bmatrix}.
\]  \hspace{1cm} (37)

So long as the system is not at the critical point (i.e., \( \omega_+ \neq \omega_- \)), the matrix is non-singular, and we can invert it to obtain \( \psi_{\pm} \):

\[
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix}
= \frac{1}{i(\omega_+ - \omega_-)}
\begin{bmatrix}
-i\omega_- x_0 - v_0 \\
i\omega_+ x_0 + v_0
\end{bmatrix}.
\]  \hspace{1cm} (38)

We can plug these coefficients back into the general solution. After some algebra, the result simplifies to

\[ z(t) = e^{-\gamma t} \left[ x_0 \cos(\Omega t) + \frac{\gamma x_0 + v_0}{\Omega} \sin(\Omega t) \right], \quad \text{where} \quad \Omega \equiv \sqrt{\omega_0^2 - \gamma^2}. \]  \hspace{1cm} (39)

For the under-damped case, \( \Omega \) is real, and this solution is consistent with the one we previously derived, except that it is now explicitly expressed in terms of our initial conditions \( x_0 \) and \( v_0 \). As for the over-damped case, we can perform the replacement

\[ \Omega \rightarrow i\Gamma = i\sqrt{\gamma^2 - \omega_0^2}. \]  \hspace{1cm} (40)

Then, using the relationships between trigonometric and hyperbolic functions, the solution can be re-written as

\[ z(t) = e^{-\gamma t} \left[ x_0 \cosh(\Gamma t) + \frac{\gamma x_0 + v_0}{\gamma} i \sinh(\Gamma t) \right], \quad \text{where} \quad \Gamma \equiv \sqrt{\omega_0^2 - \gamma^2}. \]  \hspace{1cm} (41)

\[ = \left( \frac{x_0}{2} + \frac{\gamma x_0 + v_0}{2\Gamma} \right) e^{-\gamma \Gamma t} + \left( \frac{x_0}{2} - \frac{\gamma x_0 + v_0}{2\Gamma} \right) e^{-\gamma \Gamma} t, \]  \hspace{1cm} (42)

which is again consistent with our previous result.

In either case, so long as we plug in real values for \( x_0 \) and \( v_0 \), the solution is guaranteed to be real for all \( t \). That’s to be expected, since the real solution is also one of the specific solutions for the complex harmonic oscillator equation.
4.5 Exercises

1. In the general solution for the complex damped harmonic oscillator equation, we encountered the complex frequencies

\[ \omega_\pm = -i \gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \] (43)

For fixed \( \omega_0 \) and \( \omega_0 > \gamma \), prove that \( \omega_\pm \) lie along a circular arc in the complex plane.

2. Derive the general solution for the critically-damped oscillator, as follows:

- Consider the complex ODE, in the under-damped regime \( \omega_0 > \gamma \). We previously showed that the general solution has the form

\[ z(t) = \psi_+ \exp \left[ \left( -\gamma - i \sqrt{\omega_0^2 - \gamma^2} \right) t \right] + \psi_- \exp \left[ \left( -\gamma + i \sqrt{\omega_0^2 - \gamma^2} \right) t \right] \] (44)

for some complex parameters \( \psi_+ \) and \( \psi_- \). Let us define the positive parameter \( \varepsilon = \sqrt{\omega_0^2 - \gamma^2} \). Re-write \( z(t) \) in terms of \( \gamma \) and \( \varepsilon \) (i.e., eliminating \( \omega_0 \)).
- The expression for \( z(t) \) presently contains the independent parameters \( \psi_+, \psi_- \), \( \omega_0 \), and \( \gamma \). We are free to re-define the parameters by taking

\[ \alpha = \psi_+ + \psi_- \] (45)

\[ \beta = -i \varepsilon (\psi_+ - \psi_-). \] (46)

Show that by using this redefinition, we can express \( z(t) \) using a new set of independent parameters, one of which is \( \varepsilon \).
- Taylor expand \( z(t) \) in the parameter \( \varepsilon \). Then show that in the limit \( \varepsilon \to 0 \), the re-parameterized form for \( z(t) \) reduces to the critically-coupled general solution.
- Repeat the above derivation for the critically-damped solution, but starting from the over-damped regime \( \gamma > \omega_0 \).

3. A parametric oscillator is an oscillator whose spring “constant” varies with time, as described by the ordinary differential equation

\[ \left[ \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \Omega(t)^2 \right] x(t) = 0, \quad \text{where} \quad \Omega(t) = \omega_0 \left[ 1 + \alpha \cos(2\omega_1 t) \right]. \] (47)

The term “parametric” refers to the fact that the parameter \( \Omega \), which is normally a constant, has been turned into a time-dependent quantity. Suppose that \( \omega_1 \ll \omega_0 \). Let \( x(t) \) be complex, and look for a solution of the form

\[ x(t) = \psi(t) e^{-i\omega_0 t}, \] (48)

where \( \psi(t) \) is a complex “envelope function” which varies much more slowly than the \( e^{-i\omega_0 t} \) factor. Mathematically, the slowness of the variation is described by

\[ \left| \frac{d^2 \psi}{dt^2} \right| \ll \omega_0 \left| \frac{d\psi}{dt} \right|. \] (49)

In such a situation, the second time derivative can be neglected; this is called the “slowly-varying envelope approximation”. By making this approximation, show that the parametric oscillator equation reduces to the form

\[ \frac{d}{dt} \left[ \ln(\psi) \right] = f(t), \] (50)

and find \( f(t) \). Hence, solve for \( \psi(t) \) and show that the oscillation amplitude \( |\psi(t)| \) consists of an exponential decay overlaid on a sinusoidal modulation.