Graphs and Matrices with Integral Spectrum in Some Families

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Introduction

Notation

Given a finite family of matrices $\mathcal{M}$ we denote by

$$Z(\mathcal{M}) = \#\{M \in \mathcal{M} \;:\; \mathrm{Spec}\, M \subseteq \mathbf{Z}\}$$

Given a finite family of graphs $\mathcal{G}$ we put

$$Z(\mathcal{G}) = Z(\mathcal{A}),$$

where $\mathcal{A}$ is the set of adjacency matrices of graphs $G \in \mathcal{G}$.

Families of Matrices and Graphs

We are interested in estimating $Z(\mathcal{M})$ and $Z(\mathcal{G})$ for various interesting families $\mathcal{M}$ and $\mathcal{G}$, such as

- Arbitrary matrices in a box:

  $$\mathcal{M}(K,h) = \{M = (m_{ij})_{ij=1}^{n} : |k_{ij} - m_{ij}| < h\}$$

  for a given $n \times n$ matrix $K = (k_{ij})_{ij=1}^{n}$;

- Symmetric matrices in a box:

  $$\mathcal{S}(K,h) = \{M = (m_{ij})_{ij=1}^{n} : m_{ij} = m_{ji},\ |k_{ij} - m_{ij}| < h\}$$

  for a given $n \times n$ matrix $K = (k_{ij})_{ij=1}^{n}$;

- $0,1$-matrices;

- Circulant graphs;

- Regular graphs;

- Arbitrary graphs.

Motivation

Top ten reasons to study these questions:

1. They are interesting

2. They are interesting

$\vdots \quad \vdots \quad \vdots$

9. They are interesting

10. They are related to network topologies that support perfect quantum state transfer: *Christandal, Datta, Ekert, Kay and Landahl (2004)* *Facer, Twamley and Cresser (2008)* *Godsil (2008)*

Integral Graphs

Background

The notion introduced by *Harary and Schwenk (1974)*

Several explicit constructions of integral graphs of various types:

Brouwer and Koolen (1993)
Wang, Li and Hoede (2005)
Lepović (2006)
So (2006)
Indulal and Vijayakumar (2007)
Stevanović, de Abreu, de Freitas and Del-Vecchio (2007)
Brouwer (2008)
Brouwer and Haemers (2009)
Liu and Wang (2010)
Carvalho and Rama (2010)

. . . and many more

Circulant Graphs

Let $S = \{s_1, s_2, \dots, s_k\}$ be a set of k integers with

$$1 \le s_1, s_2, \dots, s_k < n.$$

We consider only *undirected graphs* $\Longrightarrow$ S is symmetric:

$$s \in S \quad \text{iff} \quad n - s \in S.$$

A *circulant graph* $G(n; S)$ is a k-regular graph on n vertices $\{v_1, \dots, v_n\}$ such that v_i and v_j are incident whenever $i - j \in S$.

$G(n; S)$ is k-regular where $k = \#S$.

$G(n; S)$ is connected iff $\gcd(\{s : s \in S\}) = 1$.

Eigenvalues of $G(n;S)$ are explicitely given by:

$$\lambda_j = \sum_{s \in S} \exp\left(2\pi i \frac{js}{n}\right), \quad j = 1, \ldots, n,$$

and not that hard to control.

Let $\mathcal{C}_n = \{G(n;S) \ : \ \text{symmetric } S \subseteq \{1, \ldots n-1\}\}$

So (2006) :
A full description of circulant integral graphs (based on explicit formulas for the eigenvalues)

$$\Downarrow$$

$$Z(\mathcal{C}_n) \leq 2^{\tau(n)-1},$$

where $\tau(n)$ is divisors function.

Let

$\mathcal{C}^*_{n,k} = \{G(n;S) \in \mathcal{C}_n \ : \ \#S = k, \ G(n;S) \text{ connected}\}$

Saxena, Severini and Shparlinski (2007) :
For

$$n \geq \exp\left(c\sqrt{k \log\log(k+2)}\log k\right)$$

with some absolute constant $c > 0$, we have

$$Z(\mathcal{C}_{n,k}) = 0.$$

Klin and Kovács (2010)
Description of automorphism groups of integral circulant graphs.

Open Question 1 *What about more general Cayley graphs?*

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Integral Graphs with Few Cycles

Integral trees are fully classified by *Watanabe and Schwenk (1979)*

Classification of integral graphs with at most two cycles: *Omidi (2010)*

Arbitrary Graphs

Let $\mathcal{A}_n$ be the set of all adjacency matrices of graphs on n vertices.

Ahmadi, Alon, Blake and Shparlinski (2008):

$$Z(\mathcal{A}_n) \le 2^{n(n-1)/2-n/400}.$$

Not tight!

Conjecture: $Z(\mathcal{A}_n) = \exp(O(n))$.

For $n = 2^m$ any Cayley graph of $(\mathbf{Z}_2)^m$ is integral:

$$Z(\mathcal{A}_n) \ge 2^{\Omega(n)}.$$

Ideas Behind the Proof

- ‘Circular Law’

$$\Downarrow$$

- For most of adjacency matrices have most of eigenvalues are $\leq Cn^{1/2}$

$$\Downarrow$$

- Since eigenvalues are integral, at least one is of multiplicity $\geq cn^{1/2}$.

$$\Downarrow$$

- In matrix with an eigenvalue of multiplicity s there is an symmetric $s \times s$ minor defined by other entries.

$$\Downarrow$$

- Bound.

More Details

Distribution of Eigenvalues

As $A \in \mathcal{A}_n$ is symmetric, its eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

are real.

Füredi and Komlós (1981) .

Lemma 2 *For any $c > 1$ and large enough n*

$$\Pr_{A \in \mathcal{A}_n}\left[-c\sqrt{n} < \lambda_i < c\sqrt{n},\ i = 2, \ldots, n\right] \geq 1 - n^{-10}.$$

Let E_i denote the expected value of λ_i of $A \in \mathcal{A}_n$ chosen uniformly at random.

Corollary 3 *For* $i > 1$ *and large enough* n

$$|E_i| < 2\sqrt{n}$$

Proof. Using Lemma 2 with $c = 3/2$, we have that with probability $1-n^{-10}$, λ_i is at most $3\sqrt{n}/2$ and with probability n^{-10} it is at most n:

$$E_i < \left(1 - \frac{1}{n^{10}}\right)\frac{3}{2}\sqrt{n} + \frac{1}{n^{10}}n < 2\sqrt{n}$$

for large enough values of n. Similarly we have

$$E_i > \left(1 - \frac{1}{n^{10}}\right)\frac{-3}{2}\sqrt{n} + \frac{1}{n^{10}}(-n) > -2\sqrt{n}.$$

□

Let M_i denote the median of λ_i of $A \in \mathcal{A}_n$: for at least $0.5\#\mathcal{A}_n$ matrices $A \in \mathcal{A}_n$ we have $\lambda_i \geq M_i$.

Corollary 4 *We have*

$$|M_i| < 6\sqrt{n}, \qquad i = 2, \ldots, n.$$

Proof. Suppose that $M_i \geq 6\sqrt{n}$. Using Lemma 2 with $c = 3/2$ we obtain

$$E_i \geq \frac{1}{2}(6\sqrt{n}) + \frac{1}{2}\left(\frac{-3}{2}\sqrt{n}\right) + \frac{1}{n^{10}}(-n) \geq 2\sqrt{n},$$

which contradicts Cor. 3. Similarly $M_i \leq -6\sqrt{n}$.

□

Alon, Krivelevich and Vu (2002)

Lemma 5 *We have*

$$\Pr_{A \in \mathcal{A}_n}\left[|\lambda_s - M_s| > t\right] \leq 4e^{-t^2/8r^2}$$

where $r = \min\{s, n - s + 1\}$.

From Cor. 4 and Lemma 5 we derive:

Corollary 6 *We have,*

$$\Pr_{A \in \mathcal{A}_n} \left[|\lambda_i| < 7\sqrt{n},\ i = 2, \ldots, n \right] \geq 1 - 8e^{-n/32}.$$

Proof. By Cor. 4 we have

$$\begin{aligned} \Pr_{A \in \mathcal{A}_n} \left[\lambda_2 > 7\sqrt{n} \right] &= \Pr_{A \in \mathcal{A}_n} \left[\lambda_2 - 6\sqrt{n} > \sqrt{n} \right] \\ &\leq \Pr_{A \in \mathcal{A}_n} \left[\lambda_2 - M_2 > \sqrt{n} \right]. \end{aligned}$$

Applying Lemma 5 with $t = \sqrt{n}$ we have

$$\Pr_{A \in \mathcal{A}_n} \left[\lambda_2 > 7\sqrt{n} \right] \leq 4e^{-n/32}.$$

Similarly

$$\Pr_{A \in \mathcal{A}_n} \left[\lambda_n < -7\sqrt{n} \right] \leq 4e^{-n/32}.$$

□

Multiplicities of Eigenvalues

Let λ be an eigenvalue of a matrix M:

1. algebraic multiplicity is its order as a root of the characteristic polynomial of M;
2. geometric multiplicity is the rank of the null-space of $M - \lambda I$

M is symmetric:
algebraic multiplicity = geometric multiplicity

A principal submatrix of order r of $n \times n$ matrix M is a submatrix of M obtained by deleting rows $R_{i_1}, R_{i_2}, \ldots, R_{i_{n-r}}$ and columns $C_{i_1}, C_{i_2}, \ldots, C_{i_{n-r}}$ where $1 \leq i_1 < i_2 < \ldots < i_{n-r} \leq n$.

Principal submatrices of a symmetric matrix are symmetric too.

Well-known result:

Lemma 7 *A symmetric matrix M is of rank r iff M has a nonsingular principal submatrix of order r and has no larger principal submatrix which is nonsingular.*

Not so well-known result:

Lemma 8 *Let λ be a real number. Then the number $N_\lambda(n,s)$ of adjacency matrices of order n having λ as an eigenvalue of algebraic multiplicity s is at most*

$$N_\lambda(n,s) \le \binom{n}{s} 2^{n(n-1)/2 - s(s-1)/2}.$$

Concluding the Proof

By Cor. 6 the number of matrices with eigenvalue outside of $[-7\sqrt{n}, 7\sqrt{n}]$ is at most $8e^{-n/32}2^{n(n-1)/2}$.

The remaining matrices have one eigenvalue of multiplicity at least

$$t = \frac{n-1}{14\sqrt{n}+1}.$$

By Lemma 8, there are at most

$$\begin{aligned}
&\sum_{-7\sqrt{n}\le\lambda\le 7\sqrt{n}} \; \sum_{t\le s\le n} N_\lambda(n,s) \\
&\qquad \le (14\sqrt{n}+1) \sum_{t\le s\le n} \binom{n}{s} 2^{n(n-1)/2 - s(s-1)/2+1} \\
&\qquad \le (14\sqrt{n}+1) n \binom{n}{\lfloor n/2 \rfloor} 2^{n(n-1)/2 - t(t-1)/2+1}
\end{aligned}$$

graphs having an integral spectrum. □

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Regular Graphs

Dumitriu and Pal (2009)

Analogue of the result of *Alon, Krivelevich and Vu (2002)* about the distribution of eigenvalues.

Ahamdi and Shparlinski (???)

Nothing yet, but it's in our plans

Integral Matrices

Arbitrary matrices in a box

For a $n \times n$ matrix $K = (k_{ij})_{ij=1}^{n}$ we define

$$\mathcal{M}(K,h) = \{M = (m_{ij})_{ij=1}^{n} \ : \ |k_{ij} - m_{ij}| < h\}$$

Let O_n be the $n \times n$ zero matrix.

Martin and Wong (2009):

$$Z(\mathcal{M}(O_n,h)) \le h^{n^2-2+o(1)}$$

Shparlinski (2009): Improvement and generalisation:

$$Z(\mathcal{M}(K,h)) \le (h + \|K\|)h^{n^2-n+o(1)}$$

The result is based on a new upper bound

$$\#\{M \in \mathcal{M}(K,h) \ : \ \det M = 0\} \ll h^{n^2-n} \log h \quad (1)$$

(uniformly over K) applied to $K - \lambda I$ instead of K and the bound $\lambda = O(h + \|K\|)$ on eigenvalues λ of $M \in \mathcal{M}(K,h)$.

The proof of (1) is **elementary**.

Symmetric matrices in a box

For a $n \times n$ matrix $K = (k_{ij})_{ij=1}^{n}$ we define

$\mathcal{S}(K,h) = \{M = (m_{ij})_{ij=1}^{n} \ : \ m_{ij} = m_{ji},\ |k_{ij} - m_{ij}| < h\}$

Shparlinski (2009) :

$$Z(\mathcal{S}(K,h)) \le h^{n(n+1)/2-1+o(1)}$$

The bound is uniform over K.

The result is based on a new upper bound

$$\#\{S \in \mathcal{S}(K,h) \ : \ \det S = 0\} \ll h^{n(n+1)/2-1+o(1)} \qquad (2)$$

(uniformly over K) and a result of *Weyl (1912)* on the **stability** of eigenvalues of symmetric matrices:

For every eigenvalue λ of $M \in \mathcal{S}(K,h)$ there exists and an eigenvalue η of K with

$$\lambda = \eta + O(h)$$

Matrices with a given determinant

The case of $K = O_n$: very well studied

Duke, Rudnick and Sarnak (1993) for $a \neq 0$ and *Katznelson (1993)* when $a = 0$ we immediately obtain

$$\#\{M \in \mathcal{M}(O_n, h) : \det M = a\} \ll \begin{cases} h^{n^2-n}, & a \neq 0, \\ h^{n^2-n} \log h, & a = 0, \end{cases}$$

where the implied constant may depend on n.

Wigman (2005):
A variant of the result of *Katznelson (1993)* for matrices with *primitive rows*

Similarly, *Duke, Rudnick and Sarnak (1993)* for $a \neq 0$ and *Eskin and Katznelson (1994)* when $a = 0$

$$\#\{S \in \mathcal{S}(O_n, h) : \det S = a\} \ll \begin{cases} h^{n(n-1)/2}, & a \neq 0, \\ h^{n(n-1)/2} \log h, & a = 0. \end{cases}$$

In fact they all give **asymptotic formulas** but for matrices ordered w.r.t. a different norm.

Bourgain, Costelo, Tao, Vu and Wood (2006–???)

A series of results about singular $0, 1$-matrices and matrices with entries in $\{-k, \ldots, k\}$ for a fixed k and growing dimension k.

They also lead to various estimates on the number of integral matrices in some families. One example is given in *Bourgain, Vu and Wood (2009)* .

Arbitrary K:

The proof of (1) is **elementary**:

Assume that $X, Y \in \mathcal{M}(K,h)$ are obtained from an $n \times (n-1)$-matrix R by augmenting it by $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^n$:

$$X = (R|\mathbf{x}) \qquad \text{and} \qquad Y = (R|\mathbf{y})\,.$$

If $\det(X) = \det(Y)$ then putting

$$\mathbf{z} = \mathbf{x} - \mathbf{y} \qquad \text{and} \qquad Z = (R|\mathbf{z})$$

we get $\det Z = 0$ (expand Z with respect to the last column).

Therefore,

$$\begin{aligned}\#\{\mathbf{x} =&(x_1, \dots, x_n) \in \mathbf{Z}^n : \\ &\det(R|\mathbf{x}) = a,\ |k_{i,n} - x_i| < h,\ i = 1, \dots, n\} \\ \le \#\{&\mathbf{z} = (z_1, \dots, z_n) \in \mathbf{Z}^n : \\ &\qquad \det(R|\mathbf{z}) = 0,\ |z_i| < 2h,\ i = 1, \dots, n\}.\end{aligned}$$

If $\mathbf{x}_1, \dots, \mathbf{x}_J$ are the elements from set on the LHS then the vectors $\mathbf{z}_j = \mathbf{x}_1 - \mathbf{x}_j,\ j = 1, \dots, J$ are distinct and belong to the set on the RHS.

Summing this over all $(2h-1)^{n^2-n}$ integral matrices $R=(r_{ij})_{i,j=1}^{n,n-1}$ with

$$|k_{ij}-r_{ij}|<h, \qquad 1\le i\le n,\ 1\le j\le n-1,$$

we obtain

$$\#\{M\in\mathcal{M}(K,h) : \det M = a\} \\ \le \#\{M\in\mathcal{M}(K_1,2h) : \det M = 0\},$$

where K_1 is obtained from K by replacing its nth column by a zero vector.

Repeating the same argument with respect to the $(n-1)$th column of the matrix K_1, we obtain

$$\#\{M\in\mathcal{M}(K,h) : \det M = a\} \\ \le \#\{M\in\mathcal{M}(K_2,4h) : \ \det M = 0\},$$

where K_2 now has two zero column.

After n steps we arrive to

$$\#\{M\in\mathcal{M}(K,h) : \det M = a\} \\ \le \#\{M\in\mathcal{M}(K_n,2^n h) : \ \det M = 0\},$$

where $K_n = O_n$ is the zero matrix. $\square$

The above argument does not work for symmetric matrices (the structure disappears immediately).

The proof of (2) used **deep** results on integral points on surfaces due to *Browning, Heath-Brown and Salberger (2006)*

To apply we need to prove *absolute irreducibility* of the **highest form** H_k of the polynomial

$$F_K\left(\{X_{ij}\}_{1\le i\le j\le n}\right) = \det\left(\{k_{ij}+X_{ij}\}\right)_{i,j=1}^{n}$$

of degree n in $n(n+1)/2$ variables X_{ij}, $1 \le i \le j \le n$, where we also define $X_{ij} = X_{ji}$ for $i > j$:

- Remark that $H_K = F_{O_n}$.
- Then use induction.

Comments and Open Questions

A common weakness of the above results: They essentially deal with matrices having several (sometimes one!) integral eigenvalues rather than all integral eigenvalues.

Exploiting that fact may lead to an improvement.

Open Question 9 *Find a way to make use of all eigenvalues.*

A related problem:

Open Question 10 *Given a set of* n *real numbers* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *estimate the number of graphs having* Λ *as their spectrum.*

Our approaches do not work for algebraic eigenvalues of low degree.

Open Question 11 *Given a number field* $\mathbb{K}$, *estimate the number of graphs/matrices in the above families with all eigenvlues in* $\mathbb{K}$.

Open Question 12 *Given an integer* d, *estimate the number of graphs/matrices in the above families with all eigenvlues being algebraic numbers of degree at most* d.

Open Question 13 *Obtain tight uniform bounds on the number of integral matrices* $M \in \mathcal{M}(K, h)$ *and* $S \in \mathcal{S}(K, h)$ *of a given rank* r.

Open Question 14 *Obtain bounds on the number of integral matrices* $M \in \mathcal{M}(K,h)$ *and* $S \in \mathcal{S}(K,h)$ *with a characteristic polynomials of a certain type (for example, reducible over* **Z***).*

There are some results of *Rivin (2008)* but the do not give what is really expected for Question 14.

Open Question 15 *For a given matrix* $K = (k_{ij})_{i,j=1}^n$ *and* n^2 *polynomials* $f_{ij}(X) \in \mathbf{Z}[X]$*,* $i,j = 1,\ldots,n$*, obtain bounds on the number of integral matrices in the set:*

$$\left\{\left(f_{ij}(x_{ij})\right)_{i,j=1}^n \ : \ k_{ij} \le x_{ij} < k_{ij} + h\right\}.$$

Back to our Motivation

Integral graphs and perfect quantum state transfer?

– Not quite so . . .

What do we really need?

For every quadruple $\lambda_h, \lambda_i, \lambda_j, \lambda_k$ of eigenvalues (with $\lambda_j \neq \lambda_k$), we need

$$\frac{\lambda_h - \lambda_i}{\lambda_j - \lambda_k} \in \mathbb{Q}.$$

Saxena, Severini and Shparlinski (2007) :

For circulant graphs this property is essentially equivalent to integrality.

Open Question 16 *What about other graphs?*

Open Question 17 *If this property differs from integrality, can we estimate the number of such graphs in some interesting families?*