A Short Introduction to Index Coding with Side Information

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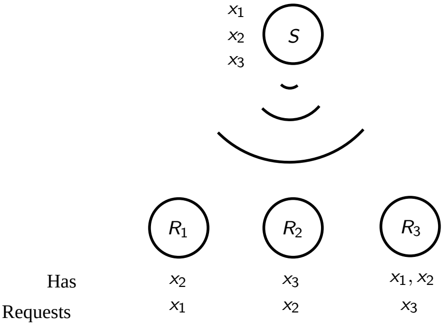
x_1
x_2
x_3
S
R_1
R_2
R_3
Has
x_2
x_3
x_1, x_2
Requests
x_1
x_2
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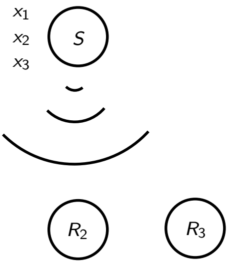

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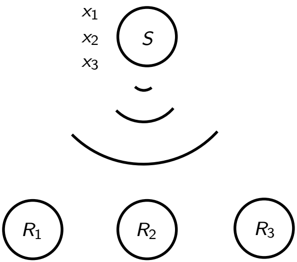
$$\begin{cases} x_1 + x_2 \\ x_2 + x_3 \end{cases}$$

Trivial solution: 3 transmissions



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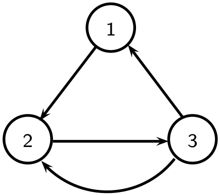

Digraph of Side Information D

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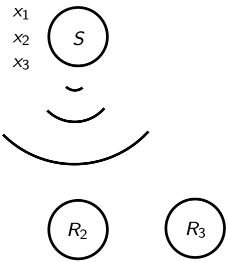

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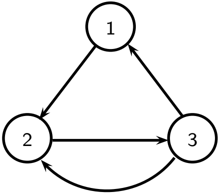

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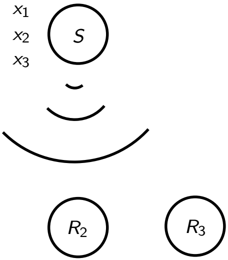
$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

has rank two. An $n \times n$ matrix $\mathbf{M}$ fits a digraph $\mathcal{D}$ of order n if

$$m_{i,j} = \begin{cases} 1, & j = i \\ 0, & (i,j) \notin \mathcal{E}(\mathcal{D}) \end{cases}$$



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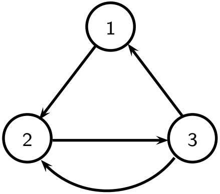

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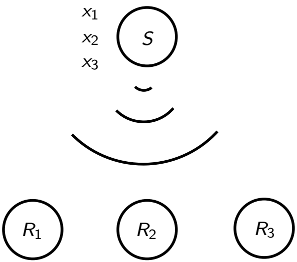

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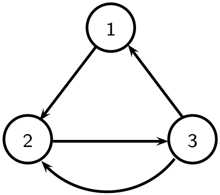

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Bar-Yossef *et al.* (2006): $1/\text{minrk}_2(\mathcal{D})$ is the best rate for scalar linear index codes (IC)

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$$\text{An IC instance} \hookrightarrow \text{Confusion graph } \mathfrak{C}$$
$$\text{Best rate} = 1/\chi(\mathfrak{C})$$

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An IC instance $\hookrightarrow$ Confusion graph $\mathfrak{C}$

Best rate $= 1/\chi(\mathfrak{C})$

El Rouayheb *et al.* (2008): block nonlinear ICs outperform block linear ICs; block linear ICs outperform scalar linear ICs

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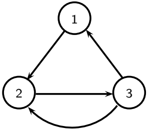

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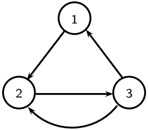

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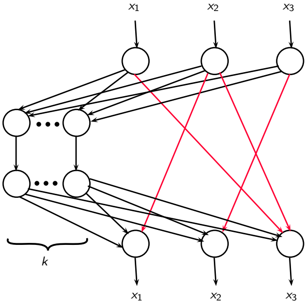

Corresponding Network Coding instance

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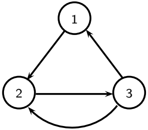

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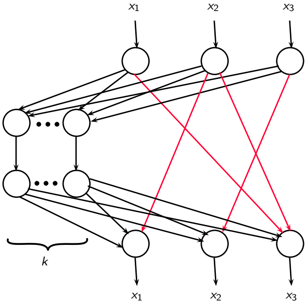

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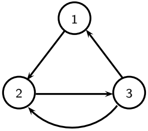

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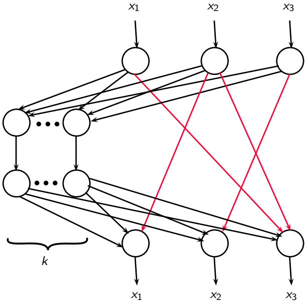

Corresponding Network Coding instance

NC can be reduced to IC

$$\text{NC instance} \hookrightarrow \text{IC instance}$$
$$\exists \text{ a linear block NC} \iff \exists \text{ a linear block IC}$$
$$\exists \text{ a nonlinear block NC} \implies \exists \text{ a nonlinear block IC}$$

Index Coding, Network Coding, and Matroid

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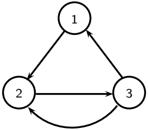

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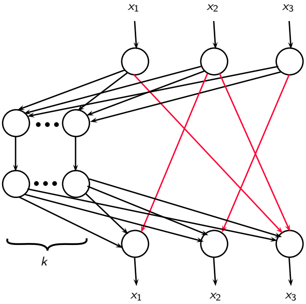

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NC instance $\hookrightarrow$ IC instance

$\exists$ a linear block NC $\Longleftrightarrow$ $\exists$ a linear block IC

$\exists$ a nonlinear block NC $\Longrightarrow$ $\exists$ a nonlinear block IC

$\downarrow$

nonlinear vs. linear results in NC

$\Downarrow$

nonlinear vs. linear results in IC

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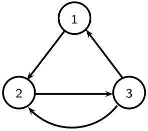

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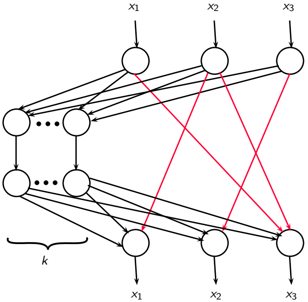

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NC can be reduced to IC

$$\begin{aligned} \text{NC instance} &\hookrightarrow \text{IC instance} \\ \exists \text{ a linear block NC} &\Longleftrightarrow \exists \text{ a linear block IC} \\ \exists \text{ a nonlinear block NC} &\Longrightarrow \exists \text{ a nonlinear block IC} \\ &\downarrow \\ \text{nonlinear vs. linear} &\text{ results in NC} \\ &\Downarrow \\ \text{nonlinear vs. linear} &\text{ results in IC} \end{aligned}$$

Matroid can be reduced to IC

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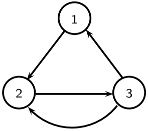

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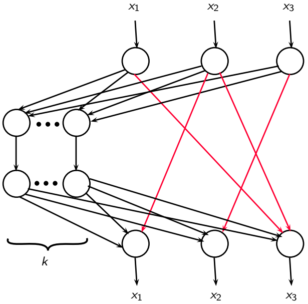

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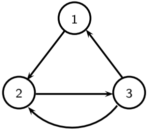
Index Coding is a special case of Network Coding (NC)



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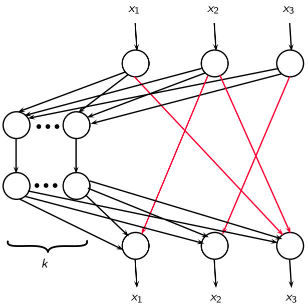

Corresponding Network Coding instance

NC can be reduced to IC

NC instance $\hookrightarrow$ IC instance

$\exists$ a linear block NC $\Longleftrightarrow$ $\exists$ a linear block IC

$\exists$ a nonlinear block NC $\Longrightarrow$ $\exists$ a nonlinear block IC

$\downarrow$

nonlinear vs. linear results in NC

$\Downarrow$

nonlinear vs. linear results in IC

Matroid can be reduced to IC

matroid $\hookrightarrow$ IC instance

$\exists$ a t-representation $\Longleftrightarrow$ $\exists$ a linear t-block IC

$\downarrow$

matroid: 2-rep. but not 1-rep.

$\Downarrow$

linear 2-block IC with better rate than linear scalar IC

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(Di)Graphs with Easy-to-Compute Capacities

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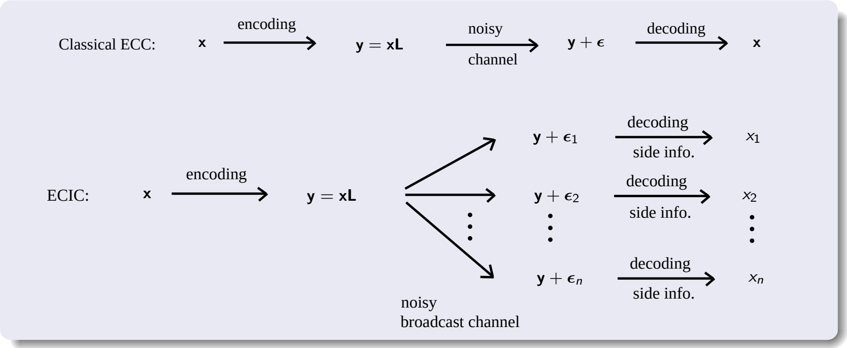
Error Correction for Index Coding

Classical ECC: $\mathbf{x} \xrightarrow{\text{encoding}} \mathbf{y} = \mathbf{x}\mathbf{L} \xrightarrow[\text{channel}]{\text{noisy}} \mathbf{y} + \epsilon \xrightarrow{\text{decoding}} \mathbf{x}$

Error Correction for Index Coding



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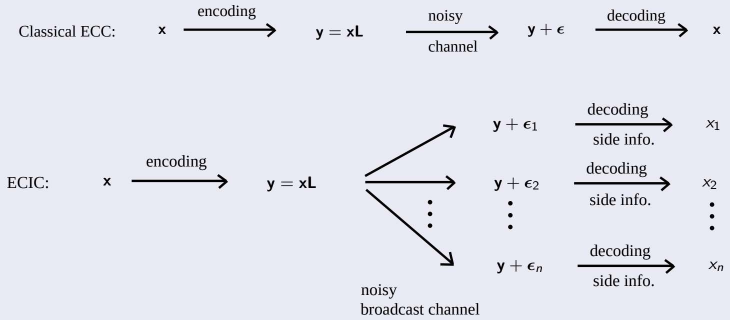

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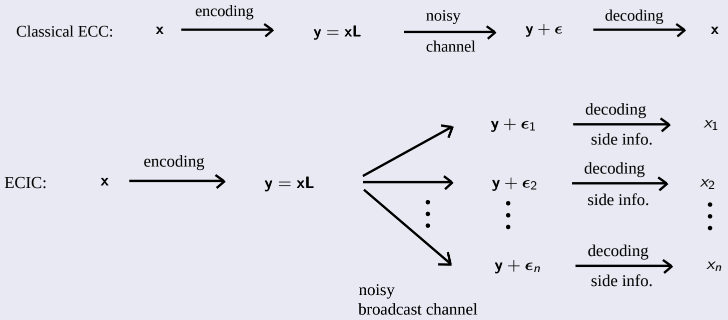

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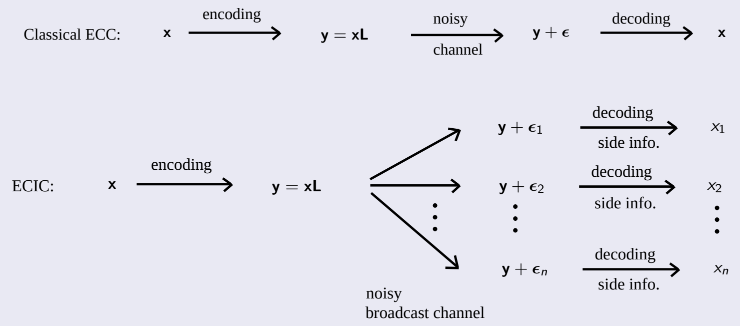

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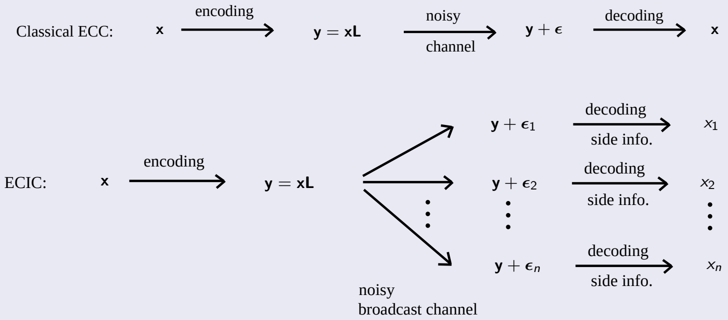

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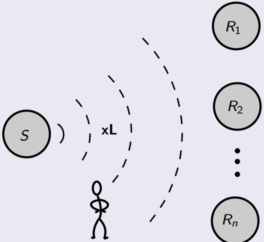

Adversary A

- owns t x_j's
- listens to μ transmissions
- introduces δ errors

GOAL:
- R_i can decode x_i
- adversary has no information about other x_j's

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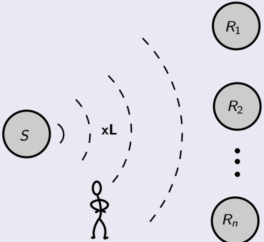

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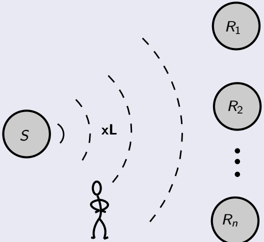

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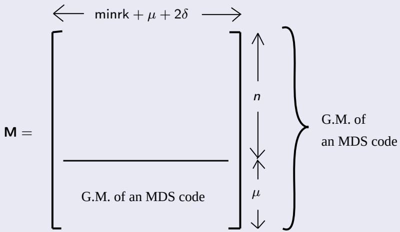

[a] introduced by Ozarow and Wyner in "Wiretap Channel II," 1985