Appendix E: Coherent States

Coherent states are special states of bosonic systems (including the quantum harmonic oscillator, whose excitation quanta can be regarded as bosonic particles) whose dynamics are highly similar to classical oscillator trajectories. They provide an important link between quantum and classical harmonic oscillators.

E.1. DEFINITION

The Hamiltonian of a simple harmonic oscillator (with $\hbar = m = \omega_0 = 1$ for simplicity) is

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}, \tag{E.1}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators. The ladder operators are

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{x} + i\hat{p}\right) \tag{E.2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{x} - i\hat{p}\right). \tag{E.3}$$

These obey the commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{E.4}$$

As a result, we can also regard these as the creation and annihilation operators for a bosonic particle that has only one single-particle state.

The Hamiltonian for the harmonic oscillator, Eq. (E.1), can be written as

$$\hat{H} = \hat{a}^\dagger\hat{a} + 1/2. \tag{E.5}$$

The annihilation operator $\hat{a}$ kills off the ground state $|\varnothing\rangle$:

$$\hat{a}|\varnothing\rangle = 0. \tag{E.6}$$

Thus, $|\varnothing\rangle$ is analogous to the "vacuum state" for a bosonic particle.

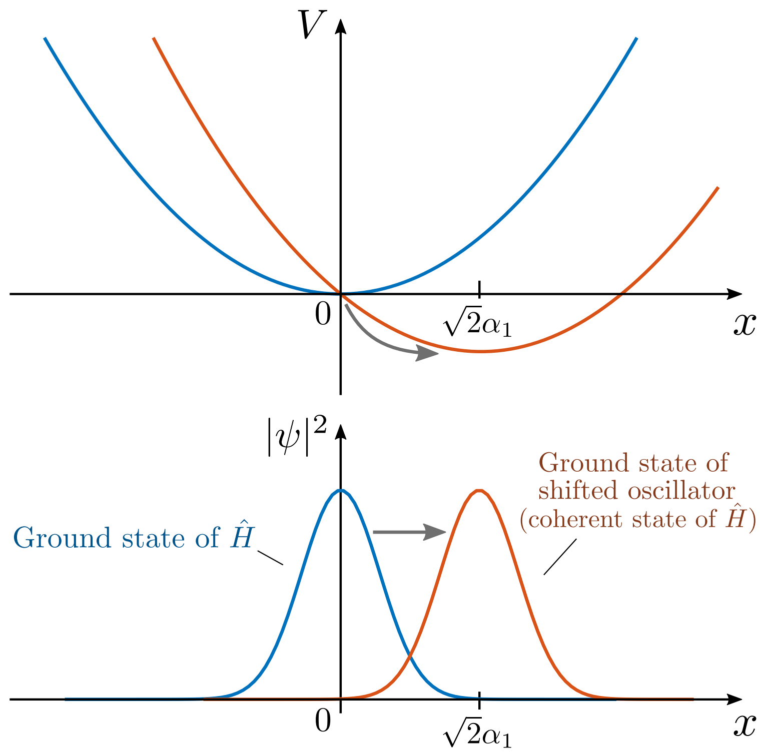
Returning to the Hamiltonian (E.1), suppose we add a term proportional to $\hat{x}$:

$$\hat{H}' = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - \sqrt{2}\alpha_1\hat{x}. \tag{E.7}$$

The coefficient of $-\sqrt{2}\alpha_1$, where $\alpha_1 \in \mathbb{R}$, is for later convenience. By completing the square, we see that this additional term corresponds to a shift in the center of the potential, plus an energy shift:

$$\hat{H}' = \frac{\hat{p}^2}{2} + \frac{1}{2}\left(\hat{x} - \sqrt{2}\alpha_1\right)^2 - \alpha_1^2. \tag{E.8}$$

Let $|\alpha_1\rangle$ denote the ground state for the shifted harmonic oscillator.



By analogy with how we solved the original harmonic oscillator problem, let us define a new annihilation operator with displaced x:

$$\hat{a}' = \frac{1}{\sqrt{2}}\left(\hat{x} - \sqrt{2}\alpha_1 + i\hat{p}\right). \tag{E.9}$$

This is related to the original annihilation operator by

$$\hat{a}' = \hat{a} - \alpha_1. \tag{E.10}$$

We can easily show that $[\hat{a}', \hat{a}'^\dagger] = 1$, and that $\hat{H}' = \hat{a}'^\dagger \hat{a}' + 1/2 - \alpha_1^2$. Hence,

$$\hat{a}' |\alpha_1\rangle = 0. \tag{E.11}$$

But Eq. (E.10) implies that in terms of the *original* annihilation operator,

$$\hat{a} |\alpha_1\rangle = \alpha_1 |\alpha_1\rangle. \tag{E.12}$$

In other words, $|\alpha_1\rangle$ is an eigenstate of the original harmonic oscillator's annihilation operator, with the displacement parameter α_1 as the corresponding eigenvalue! For reasons that will become clear later, we call $|\alpha_1\rangle$ a **coherent state** of the original harmonic oscillator $\hat{H}$.

E.2. EXPLICIT EXPRESSION FOR THE COHERENT STATE

Let us derive an explicit expression for the coherent state in terms of $\hat{a}$ and $\hat{a}^\dagger$, the creation and annihilation operators of the original harmonic oscillator. Consider the translation operator

$$\hat{T}(\Delta x) = \exp(-i\hat{p}\Delta x). \tag{E.13}$$

Since $|\alpha_1\rangle$ is the ground state of a displaced harmonic oscillator, it can be generated by performing a displacement of the original oscillator's ground state $|\varnothing\rangle$. The displacement is $\Delta x = \sqrt{2}\alpha_1$:

$$|\alpha_1\rangle = \hat{T}\big(\sqrt{2}\alpha_1\big)\,|\varnothing\rangle \tag{E.14}$$

$$= \exp\left[\alpha_1\left(\hat{a}^\dagger - \hat{a}\right)\right]|\varnothing\rangle. \tag{E.15}$$

In deriving the second line, we have used Eqs. (E.2)–(E.3) to express $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. We can further simplify the result by using the Baker-Campbell-Hausdorff formula for operator exponentials:

$$\text{If}\;\; [[\hat{A},\hat{B}],\hat{A}] = [[\hat{A},\hat{B}],\hat{B}] = 0 \;\;\Rightarrow\;\; e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2}\,e^{\hat{A}}e^{\hat{B}}. \tag{E.16}$$

The result is

$$|\alpha_1\rangle = e^{-\alpha_1^2/2}\,e^{\alpha_1\hat{a}^\dagger}|\varnothing\rangle. \tag{E.17}$$

If we write the exponential in its series form,

$$|\alpha_1\rangle = e^{-\alpha_1^2/2}\left(1 + \alpha_1\hat{a}^\dagger + \frac{\alpha_1^2}{2}\left(\hat{a}^\dagger\right)^2 + \cdots\right)|\varnothing\rangle, \tag{E.18}$$

then we see that from the point of view of the bosonic excitations of the original Hamiltonian $\hat{H}$, the state $|\alpha_1\rangle$ has an *indeterminate number of bosons*. It is a superposition of the zero-boson (vacuum) state, a one-boson state, a two-boson state, etc.

We can generalize the coherent state by performing a shift not just in space, but also in momentum. Instead of Eq. (E.7), let us define

$$\hat{H}' = \frac{1}{2}\left(\hat{p} - \sqrt{2}\alpha_2\right)^2 + \frac{1}{2}\left(\hat{x} - \sqrt{2}\alpha_1\right)^2 \tag{E.19}$$

$$= \hat{H} - \left(\alpha\hat{a}^\dagger + \alpha^*\hat{a}\right) + \text{constant}, \tag{E.20}$$

where

$$\alpha \equiv \alpha_1 + i\alpha_2 \;\in\; \mathbb{C}. \tag{E.21}$$

It can then be shown that the ground state of $\hat{H}'$, which we denote by $|\alpha\rangle$, satisfies

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle. \tag{E.22}$$

(Note that $\hat{a}$ is not Hermitian, so its eigenvalue α need not be real.) In explicit terms,

$$|\alpha\rangle \;=\; \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]|\varnothing\rangle \;=\; e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|\varnothing\rangle. \tag{E.23}$$

E.3. BASIC PROPERTIES

There is one coherent state $|\alpha\rangle$ for each complex number $\alpha \in \mathbb{C}$. They have the following properties:

1. They are normalized:

$$\langle\alpha|\alpha\rangle = 1. \tag{E.24}$$

   This follows from the fact that they are ground states of displaced harmonic oscillators.

2. They form a complete set, meaning that the identity operator can be resolved as
$$\hat{I} = C \int d^2\alpha |\alpha\rangle\langle\alpha|, \tag{E.25}$$
where C is some numerical constant and $\int d^2\alpha$ denotes an integral over the complex plane. However, the coherent states do *not* form an orthonormal set, as they are over-complete: $\langle\alpha|\alpha'\rangle \neq 0$ for $\alpha \neq \alpha'$.

3. The expected number of particles in a coherent state is
$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2. \tag{E.26}$$

4. The probability distribution of the number of particles follows a *Poisson distribution*:
$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{E.27}$$
The mean and variance of this distribution are both $|\alpha|^2$.

5. The mean position and momentum are
$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{2}\,\mathrm{Re}(\alpha) \tag{E.28}$$
$$\langle\alpha|\hat{p}|\alpha\rangle = \sqrt{2}\,\mathrm{Im}(\alpha). \tag{E.29}$$

E.4. DYNAMICAL PROPERTIES

Take the harmonic oscillator Hamiltonian with zero-point energy omitted for convenience:
$$\hat{H} = \hat{a}^\dagger\hat{a}. \tag{E.30}$$
Suppose we initialize the system in a coherent state $|\alpha_0\rangle$ for some $\alpha_0 \in \mathbb{C}$. This is not an energy eigenstate of $\hat{H}$, so how will it subsequently evolve?

It turns out that the dynamical state has the form
$$|\psi(t)\rangle = |\alpha(t)\rangle, \quad \text{where } \alpha(0) = \alpha_0. \tag{E.31}$$
In other words, the system is always in a coherent state, but the complex parameter $\alpha(t)$ varies with time. To find $\alpha(t)$, plug the ansatz into the time-dependent Schrödinger equation:
$$i\frac{d}{dt}|\alpha(t)\rangle = \hat{a}^\dagger\hat{a}|\alpha(t)\rangle \tag{E.32}$$
$$i\left\langle\alpha(t)\middle|\frac{d}{dt}\middle|\alpha(t)\right\rangle = |\alpha(t)|^2. \tag{E.33}$$
We can calculate the left-hand side using Eqs. (E.22), (E.23), and (E.24):
$$i\left\langle\alpha(t)\middle|\frac{d}{dt}\middle|\alpha(t)\right\rangle = i\,\langle\alpha(t)|\,\frac{d}{dt}\left[e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}\right]|\varnothing\rangle \tag{E.34}$$
$$= i\,\langle\alpha(t)|\left(-\frac{1}{2}\frac{d}{dt}(\alpha\alpha^*) + \frac{d\alpha}{dt}\hat{a}^\dagger\right)|\alpha(t)\rangle \tag{E.35}$$
$$= i\left(-\frac{1}{2}\dot{\alpha}\alpha^* - \frac{1}{2}\alpha\dot{\alpha}^* + \dot{\alpha}\alpha^*\right). \tag{E.36}$$

Hence,

$$\frac{i}{2}\left(\dot{\alpha}\alpha^* - \alpha\dot{\alpha}^*\right) = |\alpha|^2. \tag{E.37}$$

This looks more complicated than it actually is. Dividing both sides by $\alpha\alpha^*$ gives

$$\frac{1}{2}\left(\frac{i\dot{\alpha}}{\alpha} + \left[\frac{i\dot{\alpha}}{\alpha}\right]^*\right) = \mathrm{Re}\left[\frac{i\dot{\alpha}}{\alpha}\right] = 1. \tag{E.38}$$

This reduces to

$$\dot{\alpha} = \big[-i + \gamma(t)\big]\alpha(t), \quad \gamma \in \mathbb{R}, \tag{E.39}$$

which is the equation of motion for a complex harmonic oscillator with an arbitrary damping or amplification factor γ. For $\gamma = 0$, the oscillator is energy-conserving and the solutions are

$$\alpha(t) = \alpha_0\, e^{-it}. \tag{E.40}$$

Referring back to Eqs. (E.28)–(E.29), this implies that the mean position and momentum have the following time-dependence:

$$\langle x\rangle = \sqrt{2}|\alpha_0|\cos\left[t - \arg(\alpha_0)\right] \tag{E.41}$$

$$\langle p\rangle = -\sqrt{2}|\alpha_0|\sin\left[t - \arg(\alpha_0)\right]. \tag{E.42}$$

The dynamics of a coherent state therefore reproduces the motion of a classical harmonic oscillator with $m = \omega_0 = 1$.